

# SPLITTING METAPLECTIC SUBGROUPS

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ABSTRACT. The purpose of this note is to provide a readable and comprehensive proof of one result about splitting Metaplectic subgroups.

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## 1. STATEMENT OF THE RESULTS

Let  $F$  be a non-archimedean local field of *odd* residual characteristic, and let  $D$  be a division algebra over  $F$  with a canonical involution  $\tau$  such that  $F$  consists of all  $\tau$ -fixed points of  $D$ . We let  $(W, \langle, \rangle)$  be a symplectic space over  $F$  of dimension  $2n$  with a decomposition of tensor product

$$W = W_1 \otimes_D W_2, \quad \langle, \rangle = \text{Trd}_{D/F} (\langle, \rangle_1 \otimes \tau(\langle, \rangle_2))$$

for two  $\epsilon_i$ -hermitian spaces  $(W_1, \langle, \rangle_1)$ ,  $(W_2, \langle, \rangle_2)$  over  $D$  with  $\epsilon_1 \epsilon_2 = -1$ . We will let  $U(W_i)$  be the group of isometries of  $(W_i, \langle, \rangle_i)$ , and  $\text{GU}(W_i)$  the group of isometries of similitudes of  $(W_i, \langle, \rangle_i)$ . The

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canonical intermediate subgroup  $\Gamma$  of  $\mathrm{Sp}(W)$  is defined as

$$\Gamma := \{(g_1, g_2) | g_1 \in \mathrm{GU}(W_1), g_2 \in \mathrm{GU}(W_2) \text{ such that } \lambda_1(g_1)\lambda_2(g_2) = 1\},$$

for the similitude characters  $\lambda_i$  defined from  $\mathrm{GU}(W_i)$  to  $F^\times$ ,  $i = 1, 2$ . Accordingly there exists a canonical map

$$\iota : \Gamma \longrightarrow \mathrm{Sp}(W, \langle, \rangle)$$

with the kernel  $A_2 = \{(1, 1), (-1, -1)\}$ . Writing  $\overline{\mathrm{Sp}(W)}$  for the central topological extension of  $\mathrm{Sp}(W)$  by  $\mu_8$  (see §9), we will let  $\overline{U}(W_i)$ ,  $\overline{\Gamma}$  be the inverse images of  $U(W_i)$ ,  $\Gamma$  in  $\overline{\mathrm{Sp}(W)}$  respectively.

By [11, Page 15], it was shown that except the case  $\epsilon_1 = 1, \epsilon_2 = -1$ , and  $W_2 = \mathbb{H}$ , the pair  $(U(W_1), U(W_2))$  is the so-called irreducible *dual reductive pair* of type I in the sense of Howe. The following delicate splitting results about above irreducible dual reductive pair are coming from [11, Page 52].

**Theorem 1.1 (MVW).** *Except for  $W_1$  being symplectic and  $W_2$  being orthogonal of odd dimension, the group extension  $\overline{U}(W_1)$  is splitting over  $U(W_1)$ .*

We remark that in [9], Kudla investigated precisely the explicit splitting group extensions of dual reductive pairs. Under the same spirit, one can confer to Pan's article [14] for the latest developments. Nevertheless in this note by following [11]'s approach, from above Theorem 1.1, we derive the result:

**Theorem A.** *The exact sequence*

$$1 \longrightarrow \mu_8 \longrightarrow \overline{\Gamma} \longrightarrow \iota(\Gamma) \longrightarrow 0$$

*is splitting, except when the irreducible dual reductive pair is a symplectic-orthogonal type, and the orthogonal vector space over  $F$  is of odd dimension.*

To prove this theorem, we divide this paper essentially into two parts. In Sections 2—9, we prepare adequately known tools and statements so as to proving the main theorem more precisely and confidently. To keep the trace, we restrict ourself here to explain briefly those techniques and results built in the first part. Sections 2, 3, are devoted to a review of concrete facts concerning some specific so-called hyperbolic unitary groups over  $D$  and their commutator subgroups; there almost all results have been discussed in [4] and [20]. In next section, we recall a result of Tsukamoto (*cf.* [22]) about the classification of the skew hermitian spaces over a  $p$ -adic division algebra. After understanding the general setting of  $\epsilon$ -hermitian spaces over  $D$ , we describe the anisotropic unitary groups in more detail in Sections 5—8. To do this, we consult Riehm's paper [17] involving the fine structure of the norm one subgroup of  $p$ -adic division algebra. In Sections 7, 8, we present the adaption of some results of Satake [19] to our situation with the help of Gan and Tanton's work [7]. In Section 9, with outlining Moore's version of Hochschild-Serre spectral sequence in the measurable group cohomology [12] [13], we indicate the associated inflation-restriction exact sequence of six terms. Our main tool is built in Section 10 by proposing a criterion parameterized into four conditions. It is also interested to interpret it using much richer cohomology theory, developed for instance, in [2] and [15] [16]. In next Sections 11—17, we carry out the proofs of our main result routinely. The difficulty appears in the quaternionic cases (*cf.* §§14—16), and one may come across the familiar phenomenon in [7] or [24].

### 1.1. Notations.

- $F$ : a non-archimedean local field of odd residual characteristic;
- $E$ : a quadratic field extension of  $F$ ;
- $\mathbb{H}$ : the unique quaternion algebra over  $F$ , up to isometry;

- $\text{Nrd}, \text{Trd}$ : the reduced norm, resp. trace of  $\mathbb{H}$ ;
- $\mathbb{H}^0$ : the set of elements of pure quaternion of  $\mathbb{H}$ ;
- $\mathfrak{D}$ : the ring of integers of  $\mathbb{H}$  consisting of the elements  $\mathfrak{d} \in \mathbb{H}$  satisfying  $|\text{Nrd}(\mathfrak{d})|_F \leq 1$ ;
- $\mathfrak{P}$ : the maximal ideal of  $\mathfrak{D}$ ;
- $k_{\mathbb{H}}$ : the residue field of  $\mathbb{H}$ ;
- $(D, \tau)$ : a division algebra over  $F$  with a canonical involution  $\tau$  such that  $F$  consists of all  $\tau$ -fixed points of  $D$ ; it has one of the following forms: (1)  $D = F, \tau = \text{Id}$ ; (2)  $D = E, \tau = \text{the canonical conjugation}$ ; (3)  $D = \mathbb{H}, \tau = \text{the canonical involution}$ ;
- $(H, \langle, \rangle)$ : a right (resp. left)  $\epsilon$ -hermitian hyperbolic plane over  $D$ , defined as  $\langle (d_1, d_1^*), (d_2, d_2^*) \rangle = \tau(d_1)d_2^* + \epsilon\tau(d_1^*)d_2$ , (resp.  $\langle (d_1, d_1^*), (d_2, d_2^*) \rangle = d_1\tau(d_2^*) + \epsilon d_1^*\tau(d_2)$ ), for  $d_1, d_2, d_1^*, d_2^* \in D$ ;
- $\mu_8$ : the cyclic group of the roots of unity in  $\mathbb{C}$  of order 8, generated by  $e^{\frac{2\pi i}{8}}$ ;
- $(W, \langle, \rangle), (W_i, \langle, \rangle_i), \epsilon_i, U(W_i), \Gamma$ , and  $\overline{U}(W_i), \overline{\Gamma}, \mu_8$ , etc.: introduced above.

## 2. THE HYPERBOLIC UNITARY GROUPS

In the following two sections, we review some results of unitary groups. Our main references are Scharlau's book [20] and Hahn-O'Meara's [4].

**2.1. Hyperbolic unitary groups.** Suppose now that  $V = nH$  is a right  $\epsilon$ -hermitian hyperbolic space over  $D$ . Let  $\mathcal{A} = \{e_1, \dots, e_n; e_1^*, \dots, e_n^*\}$  be a split hyperbolic basis of  $V$  so that  $\langle e_i, e_j \rangle = 0 = \langle e_k^*, e_l^* \rangle$ ,  $\langle e_r, e_s^* \rangle = \delta_{rs}$ . Clearly  $X = \text{Span}\{e_1, \dots, e_n\}$  and  $X^* = \text{Span}\{e_1^*, \dots, e_n^*\}$  both are totally isotropic subspaces of  $V$ . We then can write each element in block form  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  with  $a \in \text{End}_D(X)$ ,  $b \in \text{Hom}_D(X^*, X)$ ,  $c \in \text{Hom}_D(X, X^*)$ ,  $d \in \text{End}_D(X^*)$ . For simplicity, under the above basis  $\mathcal{A}$ , we identify  $U(V)$  with the unitary matrix group  $U_V(D)$  which consists of all elements  $G = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_{2n}(D)$  such that

$$G^* \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix} G = \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}, \quad (2.1)$$

where  $*$  :  $\text{GL}_{2n}(D) \longrightarrow \text{GL}_{2n}(D)$  is the conjugate transpose operator. In particular, (2.1) is equivalent to, the defining equations(cf. [20, p.262]):

$$\begin{aligned} \epsilon c^* a + a^* c &= 0, \\ \epsilon d^* b + b^* d &= 0, \\ \epsilon c^* b + a^* d &= 1. \end{aligned} \quad (2.2)$$

**Lemma 2.1.** *The unitary group  $U_V(D)$  is generated by the following elements:*

- (1)  $\begin{pmatrix} a & 0 \\ 0 & (a^*)^{-1} \end{pmatrix}$ , with  $a$  invertible;
- (2)  $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$ , with  $b^* = -\epsilon b$ ;
- (3)  $\begin{pmatrix} 0 & \epsilon \cdot 1 \\ 1 & 0 \end{pmatrix}$ .

*Proof.* See [20, Page 263]. □

**2.2. The group  $\text{EU}_V(D)$  I.** Recall that the elementary linear group  $\text{E}_{2n}(D)$  is a subgroup of  $\text{GL}_{2n}(D)$  generated by all the matrices  $e_{ij}(t) = 1 + \varepsilon_{ij}(t)$  with  $i \neq j$ , where  $\varepsilon_{ij}(t)$  is the matrix with value  $t$  in the  $(i, j)$ -entry and 0 elsewhere. Notice that those  $e_{ij}(t)$  satisfy the relations (cf. Hahn-O'Meara [4, p.28]):

$$\begin{cases} e_{ij}(r)e_{ij}(s) = e_{ij}(r+s), & (E1) \\ [e_{ij}(r), e_{kl}(s)] = 1, & j \neq k, i \neq l, & (E2) \\ [e_{ij}(r), e_{jl}(s)] = e_{il}(rs), & i \neq l. & (E3) \end{cases}$$

**Lemma 2.2.** *For any  $n \geq 1$ , we have  $[\text{E}_{2n}(D), \text{E}_{2n}(D)] = \text{E}_{2n}(D)$ .*

*Proof.* See Hahn-O'Meara [4, p.28]. □

Now we let  $\text{EU}_V(D)$  be a subgroup of  $\text{U}_V(D)$  generated by the following three kinds of elements:

- (1)  $A_{ij}(r) = \begin{pmatrix} 1 + \varepsilon_{ij}(r) & 0 \\ 0 & 1 - \varepsilon_{ji}(\bar{r}) \end{pmatrix}$ , for  $1 \leq i \neq j \leq n$ ;
- (2)  $B_{ij}(s) = \begin{pmatrix} 1 & \varepsilon_{ij}(s) - \varepsilon_{ji}(\bar{s}) \\ 0 & 1 \end{pmatrix}$ , for  $1 \leq i, j \leq n$ ;
- (3)  $C_{ij}(t) = \begin{pmatrix} 1 & 0 \\ -\varepsilon_{ij}(t) + \varepsilon_{ji}(\bar{t}) & 1 \end{pmatrix}$ , for  $1 \leq i, j \leq n$ .

Here,  $\varepsilon_{ij}(\cdot)$  is the matrix of  $\text{GL}_n(D)$  with only possible non-zero value in the  $(i, j)$ -position and 0 elsewhere. Notice that  $B_{ij}(s)^* = C_{ij}(s)$ .

**Lemma 2.3.** *For  $n \geq 2$ , we have:*

- (1)  $[A_{ij}(r), A_{jl}(s)] = A_{il}(rs)$ , for  $i \neq j, j \neq l$  and  $l \neq i$ ;
- (2)  $[A_{ij}(r), B_{jl}(s)] = B_{il}(rs)$ , for  $i \neq j, j \neq l$ ;
- (3)  $[A_{ij}(r), C_{il}(t)] = C_{jl}(-\bar{r}t)$ , for  $j \neq i, i \neq l$ .

Here, the indices  $i, j, l$  all belong to  $\{1, \dots, n\}$ .

*Proof.* (1) follows from above (E3). The other ones come from direct computations. □

**Corollary 2.4.** *For  $n \geq 3$ , we have  $[\text{EU}_V(D), \text{EU}_V(D)] = \text{EU}_V(D)$ .*

**2.3. The group  $\text{EU}_V(D)$  II.** Now let us treat the case  $n = 2$  and  $\dim(V) = 4$  in details. As shown before, in this case  $\text{EU}_V(D)$  is generated by the following elements:

$$\begin{aligned} (1) \quad A_{12}(r) &= \begin{bmatrix} \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} & \\ & \begin{pmatrix} 1 & 0 \\ -\bar{r} & 1 \end{pmatrix} \end{bmatrix} \text{ and } A_{21}(r) = \begin{bmatrix} \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} & \\ & \begin{pmatrix} 1 & -\bar{r} \\ 0 & 1 \end{pmatrix} \end{bmatrix}, \text{ for } r \in D; \\ (2) \quad B_{11}(s) &= \begin{bmatrix} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} & \begin{pmatrix} s - \varepsilon\bar{s} & 0 \\ 0 & 0 \end{pmatrix} \\ \mathbf{0} & \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \end{bmatrix}, B_{12}(s) = \begin{bmatrix} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} & \begin{pmatrix} 0 & s \\ -\varepsilon\bar{s} & 0 \end{pmatrix} \\ \mathbf{0} & \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \end{bmatrix}, B_{21}(s) = \begin{bmatrix} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} & \begin{pmatrix} 0 & -\varepsilon\bar{s} \\ s & 0 \end{pmatrix} \\ \mathbf{0} & \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \end{bmatrix} \\ \text{and } B_{22}(s) &= \begin{bmatrix} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & s - \varepsilon\bar{s} \end{pmatrix} \\ \mathbf{0} & \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \end{bmatrix}, \text{ for } s \in D; \end{aligned}$$

$$(3) \quad C_{11}(t) = \begin{bmatrix} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} & \mathbf{0} \\ \begin{pmatrix} \bar{t} - \epsilon t & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \end{bmatrix}, \quad C_{12}(t) = \begin{bmatrix} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} & \mathbf{0} \\ \begin{pmatrix} 0 & -\epsilon t \\ \bar{t} & 0 \end{pmatrix} & \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \end{bmatrix}, \quad C_{21}(t) = \begin{bmatrix} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} & \mathbf{0} \\ \begin{pmatrix} 0 & \bar{t} \\ -\epsilon t & 0 \end{pmatrix} & \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \end{bmatrix}$$

$$\text{and } C_{22}(t) = \begin{bmatrix} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} & \mathbf{0} \\ \begin{pmatrix} 0 & 0 \\ 0 & \bar{t} - \epsilon t \end{pmatrix} & \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \end{bmatrix}, \text{ for } t \in D.$$

**Example 2.5.** (1)  $A_{12}(r)A_{21}(-r^{-1})A_{12}(r) = H(r)$ , where  $H(r) = \text{diag} \left[ \begin{pmatrix} 0 & r \\ -r^{-1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \bar{r}^{-1} \\ -\bar{r} & 0 \end{pmatrix} \right]$ , for  $r \in D^\times$ .

(2)  $E(r) := H(r)H(-1) = \text{diag} \left[ \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}, \begin{pmatrix} \bar{r}^{-1} & 0 \\ 0 & \bar{r} \end{pmatrix} \right]$ , for  $r \in D^\times$ .

*Proof.* (2) is a consequence of (1). Now let us check (1). By definition, we have

$$\begin{bmatrix} \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} & \\ & \begin{pmatrix} 1 & 0 \\ -\bar{r} & 1 \end{pmatrix} \end{bmatrix} \begin{bmatrix} \begin{pmatrix} 1 & 0 \\ -r^{-1} & 1 \end{pmatrix} & \\ & \begin{pmatrix} 1 & \bar{r}^{-1} \\ 0 & 1 \end{pmatrix} \end{bmatrix} \begin{bmatrix} \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} & \\ & \begin{pmatrix} 1 & 0 \\ -\bar{r} & 1 \end{pmatrix} \end{bmatrix} = H(r).$$

□

**Lemma 2.6.** (1)  $[A_{12}(r), B_{21}(s)] = B_{11}(rs)$ , and  $[A_{21}(r), B_{12}(s)] = B_{22}(rs)$ , for  $r, s \in D$ .

(2)  $[A_{12}(r), B_{22}(s)] = B_{11}(rs\bar{r})B_{12}(r(s - \epsilon\bar{s}))$ , and  $[A_{21}(r), B_{11}(s)] = B_{22}(rs\bar{r})B_{21}(r(s - \epsilon\bar{s}))$ , for  $r, s \in D$ .

(3)  $[C_{22}(t), A_{21}(r)] = C_{11}(-\bar{r}tr)C_{12}(\bar{r}(t - \epsilon\bar{t}))$ , and  $[C_{11}(t), A_{12}(r)] = C_{22}(-\bar{r}tr)C_{21}(\bar{r}(t - \epsilon\bar{t}))$ , for  $r, t \in D$ .

(4)  $[E(r), A_{12}(s)] = A_{12}(-s + rsr)$ , and  $[E(r^{-1}), A_{21}(s)] = A_{21}(-s + rsr)$ , for  $r \in D^\times, s \in D$ .

(5)  $[E(r), B_{12}(s)] = B_{12}(rs\bar{r}^{-1} - s)$ , and  $[E(r), B_{21}(s)] = B_{21}(r^{-1}s\bar{r} - s)$ , for  $r \in D^\times, s \in D$ .

*Proof.* Routine computations. □

We define a subset of  $D$  by

$$S_D = \{s - \epsilon\bar{s} \mid s \in D\} = \{b \in D \mid b^* = -\epsilon b\}.$$

**Corollary 2.7.**  $S_D = \begin{cases} 0 & \text{if } D = F, \epsilon = 1, \\ F & \text{if } D = F, \epsilon = -1, \\ F & \text{if } D = E, \epsilon = -1, \\ Fi & \text{if } D = E = F(i), \epsilon = 1, \\ F & \text{if } D \text{ is a quaternion algebra over } F, \epsilon = -1, \\ D^0 & \text{if } D \text{ is a quaternion algebra over } F, \epsilon = 1. \end{cases}$

Here,  $D^0$  consists of those elements of  $D$  with trivially reduced traces. As a consequence, except the case  $D = F, \epsilon = 1$ ,  $S_D$  contains at least one nonzero element.

By definition, in case  $D = F, \epsilon = 1$ , the group  $\text{EU}_V(D)$  is generated by the elements  $A_{12}(r), A_{21}(r')$ ,  $B_{12}(s) = B_{21}(-s)$ , and  $C_{12}(t) = C_{21}(-t)$ . In particular, there exist the following relations:

$$(O1) \quad B_{12}(s)C_{12}(-s^{-1})B_{12}(s) = L(s), \text{ where } L(s) = \begin{bmatrix} 0 & \begin{pmatrix} 0 & s \\ -s & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & s^{-1} \\ -s^{-1} & 0 \end{pmatrix} & 0 \end{bmatrix}, \text{ and } N(s) := L(-1)L(s) =$$

$$\text{diag}(s^{-1} \cdot 1_{2 \times 2}, s \cdot 1_{2 \times 2}), \text{ for } s \in D^\times;$$

$$(O2) \quad [N(r), B_{12}(s)] = B_{12}(s(r^{-2} - 1)), \text{ and } [N(r), C_{12}(t)] = C_{12}(t(r^2 - 1)), \text{ for } r \in D^\times, s, t \in D.$$

**Proposition 2.8.**  $[\text{EU}_V(D), \text{EU}_V(D)] = \text{EU}_V(D)$ .

*Proof.* 1) Let us prove the result despite of the case  $D = F$ ,  $\epsilon = 1$  at the moment. Substituting  $r = 1$  in the axiom (1) of Lemma 2.6 shows that the commutator of  $\text{EU}_V(D)$  contains  $B_{11}(s)$ ,  $B_{22}(s)$ , for any  $s \in D$ . Combing Lemma 2.6(2) with Corollary 2.7 will give the elements  $B_{12}(s)$ ,  $B_{21}(s)$ , for any  $s \in D$ . Other elements  $C_{11}(t)$ ,  $C_{12}(t)$ ,  $C_{21}(t)$  and  $C_{22}(t)$  also belong to  $[\text{EU}_V(D), \text{EU}_V(D)]$  for the same reason. For an arbitrary  $t \in D^\times$ , we choose certain  $c \in F^\times$  such that  $(ct)^2 - 1 \neq 0$ . Taking  $r = ct$ ,  $s = t((ct)^2 - 1)^{-1}$  in Lemma 2.6(4) gives  $A_{12}(t)$ ,  $A_{21}(t)$ .

2) In case  $D = F$ , and  $\epsilon = 1$ , as shown above, all elements  $A_{12}(r)$ ,  $A_{21}(r')$ ,  $B_{12}(s) = B_{21}(-s)$ ,  $C_{12}(t) = C_{21}(-t)$  lie in  $[\text{EU}_V(D), \text{EU}_V(D)]$  by Lemma 2.6 (4) and above (O 2).  $\square$

**2.4. The group  $\text{EU}_V(D)$  II.** Now let us treat the case  $n = 1$  and  $\dim(V) = 2$  in details, in which case  $\text{EU}_V(D)$  is generated by the following elements:  $B(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ , and  $C(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ , for  $s, t \in \mathcal{S}_D = \{b \in D | b^* = -\epsilon b\}$ .

**Lemma 2.9.**  $\text{EU}_V(D) = [\text{EU}_V(D), \text{EU}_V(D)]$ .

*Proof.* The case  $D = F$ ,  $\epsilon = 1$  is trivial. For the other cases,  $\mathcal{S}_D$  contains non-zero elements. For any  $s \in \mathcal{S}_D^\times$  we have  $H(s) := B(s)C(-s^{-1})B(s) = \begin{pmatrix} 0 & s \\ -s^{-1} & 0 \end{pmatrix}$ ; we take one  $s_0 \in \mathcal{S}_D^\times$  and write  $E(s) := H(s)H(-s_0) = \begin{pmatrix} ss_0^{-1} & 0 \\ 0 & s^{-1}s_0 \end{pmatrix}$ . Then easy computations give  $[E(r), B(s)] = B(rs_0^{-1}ss_0^{-1}r - s)$ , and  $[E(r^{-1}), C(t)] = C(rs_0ts_0r - t)$ , for  $r \in \mathcal{S}_D^\times$ ,  $s, t \in \mathcal{S}_D$ . Using the same technique as the proof of Proposition 2.8 we obtain the result.  $\square$

$$\text{Now let } \mathfrak{A} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in D^\times \right\}.$$

**Remark 2.10.** Despite of the case  $D = F$ ,  $\epsilon = 1$ , in Lemma 2.1 the axiom (3) can deduce from there (1), (2).

*Proof.* By Corollary 2.7, we take a nonzero element  $c \in \mathcal{S}_D$ , so that  $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \begin{pmatrix} \bar{c}^{-1} & 1 \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & c^{-1} \\ 0 & 1 \end{pmatrix}$ , which gives the result.  $\square$

**Lemma 2.11.** Despite of the case  $D = F$ ,  $\epsilon = 1$ ,  $\text{U}_V(D) = \mathfrak{A} \cdot \text{EU}_V(D)$ .

*Proof.* By above proof, the group  $\text{EU}_V(D)$  contains the classes of generators in (2) of Lemma 2.1, and then the result is clear.  $\square$

**Proposition 2.12.** Despite of the case  $D = F$ ,  $\epsilon = 1$ , there is a surjective map  $\frac{D^\times}{[D^\times, D^\times]} \longrightarrow \frac{\text{U}_V(D)}{[\text{U}_V(D), \text{U}_V(D)]}$ .

*Proof.* It is immediate from Lemmas 2.9, 2.11.  $\square$

**Lemma 2.13.** In case  $D = F$ ,  $\epsilon = 1$ ,

- (1)  $\text{EU}_V(D) = 1$ , and  $[\text{U}_V(D), \text{U}_V(D)] = \left\{ \begin{pmatrix} a^2 & 0 \\ 0 & a^{-2} \end{pmatrix} \mid a \in F^\times \right\}$ .
- (2)  $\text{U}_V(D)/[\text{U}_V(D), \text{U}_V(D)] \simeq F^\times/(F^\times)^2 \times \mathbb{Z}_2$ .

*Proof.* It follows from the definition.  $\square$

**2.5. The commutator of  $\text{U}_V(D)$ .** Let  $V_1$  be a vector subspace of  $V$  generated by the elements  $e_1, e_1^*$  of  $\mathcal{A}$  so that we can embed  $\text{U}_{V_1}(D)$  into  $\text{U}_V(D)$  by

$$\epsilon_{11}(a) + \epsilon_{12}(b) + \epsilon_{21}(c) + \epsilon_{22}(d) \longrightarrow 1 + \epsilon_{11}(a - 1) + \epsilon_{1,n+1}(b) + \epsilon_{n+1,1}(c) + \epsilon_{n+1,n+1}(d - 1).$$

**Theorem 2.14** ([4, Theorem, p.230]).  $\text{U}_V(D) = \text{U}_{V_1}(D) \text{EU}_V(D)$

*Proof.* We follow the proof of [4]. The group  $\text{U}_{V_1}(D)$  is generated by  $A(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ ,  $B(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ ,  $C(b) = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}$  for  $a, \bar{b} = -\epsilon b$  in  $D$ , and  $E(\epsilon) = \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}$ . Let us firstly show that  $\text{U}_{V_1}(D)$  normalizes  $\text{EU}_V(D)$  which is generated by the elements  $A_{ij}(r)$ ,  $B_{ij}(s)$ , and  $C_{ij}(t)$  given at the beginning of §2.2. Easy computations give the following relations:

- (1)  $A(a)A_{ij}(r)A(a^{-1}) = \begin{cases} A_{ij}(ar) & \text{if } i = 1 \text{ and } i \neq j, \\ A_{ij}(ra^{-1}) & \text{if } j = 1 \text{ and } i \neq j, \\ A_{ij}(r) & \text{if } i \neq 1, j \neq 1 \text{ and } i \neq j. \end{cases}$
- (2)  $A(a)B_{ij}(s)A(a^{-1}) = \begin{cases} B_{ij}(as\bar{a}) & \text{if } i = j = 1, \\ B_{ij}(as) & \text{if } i = 1, j \neq 1, \\ B_{ij}(s\bar{a}) & \text{if } i \neq 1, j = 1, \\ B_{ij}(s) & \text{if } i \neq 1, j \neq 1. \end{cases}$
- (3)  $A(a)C_{ij}(t)A(a^{-1}) = A(a)B_{ij}(t)^*A(a^{-1}) = A(\bar{a})^*B_{ij}(t)^*A(\bar{a}^{-1})^* = (A(\bar{a}^{-1})B_{ij}(t)A(\bar{a}))^* = C_{ij}(x)$ , for some  $x \in D$ .
- (4)  $E(\epsilon)A_{ij}(r)E(\epsilon)^{-1} = \begin{cases} C_{ij}(-\epsilon r) & \text{if } i = 1, j \neq 1, \\ B_{ij}(r) & \text{if } i \neq 1, j = 1, \\ A_{ij}(r) & \text{if } i \neq 1, j \neq 1. \end{cases}$
- (5)  $E(\epsilon)B_{ij}(s)E(\epsilon)^{-1} = \begin{cases} A_{ji}(-\bar{s}) & \text{if } i = 1, j \neq 1, \\ A_{ij}(\epsilon s) & \text{if } i \neq 1, j = 1, \\ C_{ij}(-s) & \text{if } i = 1, j = 1, \\ B_{ij}(s) & \text{if } i \neq 1, j \neq 1. \end{cases}$
- (6)  $E(\epsilon)C_{ij}(s)E(\epsilon)^{-1} = (E(\epsilon)B_{ij}(s)E(\epsilon)^{-1})^* \in \text{EU}_V(D)$ .

Note that any  $b \in D$ , satisfying  $\bar{b} = -\epsilon b$ , may be expressed in terms of the form  $b = \frac{1}{2}(b - \epsilon \bar{b})$ . Therefore  $B(b), C(b)$  belong to  $\text{EU}_V(D)$ . Secondly, we prove the result by induction on the dimension of  $V$ , and may assume that  $n \geq 2$ . By Lemma 2.1, the group  $\text{U}_V(D)$  is generated by the kinds of elements  $\begin{pmatrix} a & 0 \\ 0 & a^{*-1} \end{pmatrix}$  with invertible  $a \in \text{M}_n(D)$ ,  $\begin{pmatrix} 0 & \epsilon \cdot 1 \\ 1 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$  with  $b^* = -\epsilon b \in \text{M}_n(D)$ .

By observation, the last two kinds of elements belong to  $\text{EU}_V(D)$ . For  $T(a) = \begin{pmatrix} a & 0 \\ 0 & a^{*-1} \end{pmatrix}$  with  $a = (a_{ij})_{1 \leq i, j \leq n}$ , we let  $r$  be a nonzero element in  $\mathfrak{R} = \{a_{nj}, a_{in} \mid 1 \leq i, j \leq n\}$  such that  $|\text{Nrd}_{D/F}(r)|_F = \min\{|\text{Nrd}(x)|_F \mid 0 \neq x \in \mathfrak{R}\}$ . By multiplicity of proper elements  $A_{in}(s), A_{nj}(t) \in \text{EU}_V(D)$  on both sides of  $T(a)$ , step by step, we assume simply that (1)  $a_{nn} = r$ , and (2)  $a_{nj} = 0$ , for  $1 \leq j < n$ ; in other words,

$T(a)$  has the form:  $T(a) = \text{diag} \left( \begin{bmatrix} * & \cdots & * & * \\ \vdots & \ddots & \vdots & \vdots \\ * & * & * & * \\ 0 & 0 & 0 & r \end{bmatrix}, \begin{bmatrix} * & \cdots & * & * \\ \vdots & \ddots & \vdots & \vdots \\ * & * & * & * \\ 0 & 0 & 0 & \bar{r}^{-1} \end{bmatrix} \right)$ . By Example 2.5, the element

$$I(r) = \text{diag} \left( \begin{bmatrix} 1_{(n-2) \times (n-2)} & \\ & \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \end{bmatrix}, \begin{bmatrix} 1_{(n-2) \times (n-2)} & \\ & \begin{pmatrix} \bar{r}^{-1} & 0 \\ 0 & \bar{r} \end{pmatrix} \end{bmatrix} \right)$$

belongs to  $\text{EU}_V(D)$ , so that the  $(n, n)$ -entry of  $T(a)I(r)$  is just 1. Thus multiplying  $T(a)I(r)$  by some

suitable  $A_{in}(x)$ 's on the left-hand side can give an element of the form:  $\text{diag} \left( \begin{bmatrix} * & \cdots & * & 0 \\ \vdots & \ddots & \vdots & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} * & \cdots & * & 0 \\ \vdots & \ddots & \vdots & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)$ ,

which belongs to  $\text{U}_{V'}(D)$ , for a  $(n-1)$ -dimensional hyperbolic subspace  $V'$  of  $V$  and it achieves our aim by reducing to the strictly smaller dimensional case. Let us now deal with the matrix  $\begin{pmatrix} 0 & \epsilon \cdot 1_{n \times n} \\ 1_{n \times n} & 0 \end{pmatrix}$

analogously. We denote the image of  $\begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix} \in \text{U}_{V_1}(D)$  in  $\text{EU}_V(D)$  by  $E(\epsilon)$ . According to Example 2.5 (1), the following element

$$H(1) := 1_{2n, 2n} + \epsilon_{11}(-1) + \epsilon_{1n}(1) + \epsilon_{n1}(-1) + \epsilon_{nn}(-1) + \epsilon_{n+1, n+1}(-1) + \epsilon_{n+1, 2n}(1) + \epsilon_{2n, 1}(-1) + \epsilon_{2n, 2n}(-1)$$

belongs to  $\text{EU}_V(D)$ . By calculation,  $M(\epsilon) := H(1)^{-1}E(\epsilon)H(1) = 1_{2n, 2n} + \epsilon_{nn}(-1) + \epsilon_{n, 2n}(\epsilon) + \epsilon_{2n, n}(1) + \epsilon_{2n, 2n}(-1)$ , and

$$M(\epsilon)^{-1} \cdot \begin{pmatrix} 0 & \epsilon \cdot 1_{n \times n} \\ 1_{n \times n} & 0 \end{pmatrix} = \begin{pmatrix} \begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ 1 & & & 1 \end{bmatrix} & \begin{bmatrix} \epsilon & & & \\ & \ddots & & \\ & & \epsilon & \\ 0 & & & 0 \end{bmatrix} \\ \begin{bmatrix} & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{bmatrix} & \begin{bmatrix} & & & \\ & \ddots & & \\ & & 0 & \\ & & & 1 \end{bmatrix} \end{pmatrix}$$

which also belongs to above  $\text{U}_{V'}(D)$ , so the theorem is proved by induction on  $\dim(V)$ .  $\square$

**Proposition 2.15.** *For  $n \geq 1$ , there is a canonical surjective map  $\frac{\text{U}_{V_1}(D)}{[\text{U}_{V_1}(D), \text{U}_{V_1}(D)]} \longrightarrow \frac{\text{U}_V(D)}{[\text{U}_V(D), \text{U}_V(D)]}$ .*

*Proof.* By [4, p.13],  $[\text{U}_V(D), \text{U}_V(D)] = [\text{U}_{V_1}(D), \text{U}_{V_1}(D)][\text{U}_V(D), \text{EU}_V(D)] \supseteq [\text{U}_{V_1}(D), \text{U}_{V_1}(D)]\text{EU}_V(D)$ , so the result follows.  $\square$

### 3. THE ALMOST HYPERBOLIC UNITARY GROUPS

In this section, for convenient use, we reorganize some results of unitary groups in [4]. We will let  $(V = H \oplus V_1, \langle, \rangle)$  be a right  $\epsilon$ -hermitian space over  $D$  of dimension  $n$  with a subspace  $V_1$  and a hyperbolic plane  $H$ . For any  $\sigma \in \text{U}_V(D)$ , we define the *residual space* of  $\sigma$  by

$$\mathcal{R}_\sigma = \{(\sigma - 1)x | x \in V\},$$



equipped with a *well-defined* semi-linear form

$$(\cdot, \cdot)_\sigma : \mathcal{R}_\sigma \times \mathcal{R}_\sigma \longrightarrow D; (x, y) \longmapsto \langle x, y_1 \rangle,$$

for arbitrary  $y_1 \in D$  satisfying  $(\sigma - 1)(y_1) = y$ , so that  $(xd, yd')_\sigma = \bar{d}(x, y)_\sigma d'$ , for  $d, d' \in D$ . We call  $\sigma$  an *one-dimensional transformation* if  $\dim(\mathcal{R}_\sigma) = 1$ . It can be shown (see [4, Page 212, 5.2.7]) that the orthogonal complement of  $\mathcal{R}_\sigma$  in  $V$  is just the set

$$\mathcal{F}_\sigma = \{x \in V | \sigma(x) = x\}.$$

Mention that  $\mathcal{R}_\sigma \cap \mathcal{F}_\sigma \neq 0$  unless  $\langle \cdot, \cdot \rangle_{|\mathcal{R}_\sigma}$  is non-degenerate. Recall  $\mathcal{S}_D = \{d \in D | \bar{d} = -\epsilon d\}$ .

**Example 3.1.** (1) If  $v \in V$  is isotropic, the one-dimensional transformation  $\tau_{v,d} \in U_V(D)$ , for any  $d \in \mathcal{S}_D$ , defined as

$$\tau_{v,d}(x) = x - v\langle v \cdot d, x \rangle, \quad x \in V$$

is called an isotropic transvection.

(2) If  $v \in V$  is anisotropic such that the one-dimensional transformation  $\tau_{v,d}$  defined by

$$\tau_{v,d}(x) = x - v\langle v \cdot d, x \rangle, \quad x \in V$$

belongs to  $U_V(D)$ , then we call  $\tau_{v,d}$  a symmetry.

The vector space  $(\mathcal{R}_\sigma, (\cdot, \cdot)_\sigma)$  has the following properties (cf. [4, Page 312]):

- (R1) Above semi-linear form  $(\cdot, \cdot)_\sigma$  on  $\mathcal{R}_\sigma$  is non-degenerate;
- (R2)  $(x, y)_\sigma + \epsilon \overline{(y, x)_\sigma} = -\langle x, y \rangle$ , and  $(x, \sigma(y))_\sigma + \epsilon \overline{(y, x)_\sigma} = 0$ , for  $x, y \in \mathcal{R}_\sigma$ ;
- (R3)  $\sigma : \mathcal{R}_\sigma \longrightarrow \mathcal{R}_\sigma$  preserves  $(\cdot, \cdot)_\sigma$ .

**Proposition 3.2** ([4, Page 312]). Let  $W$  be a subspace of  $V$ , equipped with a non-degenerate semi-linear form  $(\cdot, \cdot)$  such that

$$(x, y) + \epsilon \overline{(y, x)} = -\langle x, y \rangle.$$

Then there is a unique  $\sigma \in U_V(D)$  such that  $\mathcal{R}_\sigma = W$ , and  $(\cdot, \cdot)_\sigma = (\cdot, \cdot)$ .

*Proof.* For an arbitrary  $y \in V$ , by hypothesis the mapping  $\langle \cdot, y \rangle$  from  $V$  to  $D$ , restricted to  $W$ , introduces a unique element, say  $\sigma'_y \in W$ , such that  $\langle \cdot, y \rangle|_W = (\cdot, \sigma'_y)$ . Obviously  $\sigma' : V \longrightarrow W$  defined by  $y \longmapsto \sigma'_y$ , is  $D$ -linear;  $\sigma'_y = 0$  iff  $\langle \cdot, y \rangle|_W$  is null, so  $y \in W^\perp$ , which implies that  $\sigma'$  is surjective. Now let us define  $\sigma : V \longrightarrow V$  by  $\sigma = \sigma' + 1_V$ . Then

$$\begin{aligned} \langle \sigma(x), \sigma(y) \rangle &= \langle \sigma'(x), \sigma'(y) \rangle + \langle \sigma'(x), y \rangle + \langle x, \sigma'(y) \rangle + \langle x, y \rangle \\ &= \langle \sigma'(x), \sigma'(y) \rangle + (\sigma'(x), \sigma'(y)) + \overline{\epsilon(\sigma'(y), \sigma'(x))} + \langle x, y \rangle = \langle x, y \rangle; \end{aligned}$$

clearly  $\sigma(x) = 0$  only if  $\langle x, y \rangle = \langle \sigma(x), \sigma(y) \rangle = 0$  for all  $y \in Y$  so that  $\sigma = 0$ , and consequently we can assert  $\sigma \in U_V(D)$ . The uniqueness follows from the second part of (R2) and above argument  $V = \mathcal{R}_\sigma \perp \mathcal{F}_\sigma$ .  $\square$

Now let  $\sigma \in U_V(D)$ , and  $W$  a non-degenerate subspace of  $\mathcal{R}_\sigma$  with a right orthogonal complement  $W^\perp$  in  $\mathcal{R}_\sigma$  such that  $\mathcal{R}_\sigma = W \oplus W^\perp$ .

**Lemma 3.3** ([4, Page 313]). If there is  $\sigma_1 \in U_V(D)$ ,  $\sigma_2 \in U_V(D)$  such that  $\mathcal{R}_{\sigma_1} = W$ ,  $\mathcal{R}_{\sigma_2} = W^\perp$ , and both  $\mathcal{R}_{\sigma_1}$ ,  $\mathcal{R}_{\sigma_2}$  are non-degenerate subspaces of  $\mathcal{R}_\sigma$ , then  $\sigma = \sigma_1 \circ \sigma_2$ .

*Proof.* Let  $\sigma' = \sigma - 1_V$ ,  $\sigma'_i = \sigma_i - 1_V$ , for  $i = 1, 2$ ; it reduces to show that  $\sigma' = \sigma'_1 \circ \sigma'_2 + \sigma'_1 + \sigma'_2$ . Note that according to [4, Page 312, 6.2.9] the restriction of  $(, )_\sigma$  to  $\mathcal{R}_{\sigma_i}$  is equal to  $(, )_{\sigma_i}^1$ . Taking  $x \in V$ ,  $y = w + w^\perp \in \mathcal{R}_\sigma$  with  $w \in W$ ,  $w^\perp \in W^\perp$ , we then have

$$(y, \sigma'_1 \circ \sigma'_2(x) + \sigma'_1(x) + \sigma'_2(x))_\sigma = (w, \sigma'_1(x))_\sigma + (w^\perp, \sigma'_2(x))_\sigma = \langle w, x \rangle + \langle w^\perp, x \rangle = \langle y, x \rangle = (y, \sigma'(x))_\sigma.$$

Since  $(, )_\sigma$  is non-degenerate, the result follows.  $\square$

**Lemma 3.4** ([4, Page 310]). *Let  $L = \{x_0 d \mid d \in D\}$  be a line in  $V$ .*

- (1) *If  $x_0$  is anisotropic, then there is a  $\sigma \in \text{U}_V(D)$  such that  $\mathcal{R}_\sigma = L$ .*
- (2) *If  $x_0$  is isotropic, despite of the case  $D = F$ ,  $\epsilon = 1$ , there is a  $\sigma \in \text{U}_V(D)$  such that  $\mathcal{R}_\sigma = L$ .*

*Proof.* 1) If  $x_0$  is anisotropic, and  $\langle x_0, x_0 \rangle = s_0$  such that  $\overline{s_0} = \epsilon s_0$ , we then can let  $\sigma = \tau_{x_0, r}$ , for  $r = 2\overline{s_0}^{-1} = 2(\epsilon s_0^{-1})$ , defined as  $\tau_{x_0, r}(x) = x - x_0 \langle x_0 \cdot r, x \rangle$ , for  $x \in V$ , because in this case,

$$\langle \tau_{x_0, r}(x), \tau_{x_0, r}(y) \rangle = \langle x - x_0 \langle x_0 \cdot r, x \rangle, y - x_0 \langle x_0 \cdot r, y \rangle \rangle = \langle x, y \rangle + \langle x, x_0 \rangle [-\bar{r} - \epsilon r + \epsilon r s_0 \bar{r}] \langle x_0, y \rangle = \langle x, y \rangle.$$

2) Under the hypothesis, notice that  $\mathcal{S}_D \neq 0$ . As  $x_0$  is isotropic, we can find some  $x_1 \in V$  such that  $\langle x_0, x_1 \rangle = s \in \mathcal{S}_D$ . Then the isotropic transvection  $\tau_{x_0, \bar{s}^{-1}}$  satisfies the desired condition.  $\square$

**Lemma 3.5** ([4, Page 296]). *If there is an anisotropic element in  $(\mathcal{R}_\sigma, (, )_\sigma)$ , then  $\mathcal{R}_\sigma$  has a right orthogonal decomposition*

$$\mathcal{R}_\sigma = \langle x_1 \rangle \perp \langle x_2 \rangle \perp \cdots \perp \langle x_m \rangle$$

*with  $(x_i, x_i)_\sigma \neq 0$ . Here, “right” means  $(x_i, x_j)_\sigma = 0$ , for  $1 \leq i < j \leq m$ .*

*Proof.* The proof proceeds by induction on  $\dim(\mathcal{R}_\sigma)$ , supposed to be  $m$ . The case  $m = 1$  being trivial, we can assume that  $m$  is bigger than 2. Let  $x$  be an anisotropic element in  $\mathcal{R}_\sigma$  with  $(x, x)_\sigma = t$ . By induction, it suffices to find two anisotropic elements  $x_1, x_2$  in  $V$  such that  $(x_1, x_1)_\sigma \neq 0$ ,  $(x_2, x_2)_\sigma \neq 0$ , and  $(x_1, x_2)_\sigma = 0$ . Suppose now that the restriction of  $(, )_\sigma$  on  $\langle x \rangle^\perp$  is alternating. By [4, Pages 296, 6.1.5], we have  $\dim(\langle x \rangle^\perp) = \dim({}^\perp \langle x \rangle) = m - 1$ . Notice that the case  $m = 2$  is impossible. Otherwise  $\dim(\langle x \rangle^\perp) = 1$ , and  $\langle x \rangle^\perp \perp \mathcal{R}_\sigma$ . Therefore we can choose  $0 \neq y \in (\langle x \rangle^\perp \cap {}^\perp \langle x \rangle)$ ,  $z \in \langle x \rangle^\perp$  such that  $(z, x)_\sigma = s \neq t$ ,  $(y, z)_\sigma = 1$ . Put  $x_1 = x + y\bar{t}$ ,  $x_2 = x - z$ . Then  $(x_1, x_1)_\sigma = (x + y\bar{t}, x + y\bar{t})_\sigma = t \neq 0$ ,  $(x_2, x_2)_\sigma = (x - z, x - z)_\sigma = t - s \neq 0$ , and  $(x_1, x_2)_\sigma = (x + y\bar{t}, x - z)_\sigma = t - t(y, z)_\sigma = 0$ . Suppose that the restriction of  $(, )_\sigma$  on  $\langle x \rangle^\perp$  is not alternating. We choose  $y \in \langle x \rangle^\perp$  such that  $(y, y)_\sigma \neq 0$ . Then the pair  $(x, y)$  satisfies the desired condition.  $\square$

Now we recall two results together with the proofs from [4, Pages 318-319].

**Proposition 3.6.** *In case  $D = F$ ,  $\epsilon = -1$ ,  $V$  is a symplectic vector space over  $F$ . Then  $\sigma$  is a product of finite number of isotropic transvections.*

*Proof.* By Proposition 3.2, Lemmas 3.3, 3.4, it suffices to find a decomposition of  $\mathcal{R}_\sigma$  as in Lemma 3.5, so we are reduced to assuming that  $(, )_\sigma$  on  $\mathcal{R}_\sigma$  is alternating. For a nontrivial  $x_1 \in \mathcal{R}_\sigma$ , by Lemma 3.4, we construct a  $\sigma_1 \in \text{U}_V(D)$  such that  $\mathcal{R}_{\sigma_1} = \langle x_1 \rangle$ . Since  $\dim(\mathcal{R}_{\sigma_1}) = 1$ , we can choose a  $x \in V$  such that  $\langle \sigma(x), (\sigma_1 - 1_V)(x) \rangle \neq 0$ , so  $\langle \sigma_1^{-1} \circ \sigma(x), x \rangle \neq 0$ . Applying the foremost claim to  $\sigma_1^{-1} \circ \sigma$  gives the result.  $\square$

**Proposition 3.7.** *When  $V$  is not a symplectic vector space over  $F$ ,  $\sigma$  is a product of finite number of symmetries.*

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<sup>1</sup>For an arbitrary  $y \in \mathcal{R}_{\sigma_i}$ , by (R1), there is a unique  $y' \in \mathcal{R}_{\sigma_i}$  such that  $(y, -)_\sigma|_{\mathcal{R}_{\sigma_i}} = (y', -)_{\sigma_i}$ ; applying (R3), (R2), we obtain  $\overline{\epsilon(y, -)_\sigma|_{\mathcal{R}_{\sigma_i}}} = \overline{\epsilon(y', -)_{\sigma_i}}$ . Applying (R2) again, shows that  $y - y' \in \mathcal{R}_{\sigma_i}^\perp \cap \mathcal{R}_{\sigma_i} = 0$ .

*Proof.* We follow the proof of [4, Page 319].

(1) Suppose that  $\mathcal{R}_\sigma$  contains at least an anisotropic element  $x_1 \in V$  with respect to  $\langle, \rangle$ , so that  $\langle x_1, x_1 \rangle \neq 0$ . The case  $\dim(\mathcal{R}_\sigma) = 1$  being trivial, we assume that  $\dim(\mathcal{R}_\sigma) \geq 2$ . By (R2),  $(x_1, x_1)_\sigma$  is also nonzero. Applying Lemma 3.5 to  $\mathcal{R}_\sigma$  shows that

$$\mathcal{R}_\sigma = \langle x_1 \rangle \perp \cdots \perp \langle x_m \rangle$$

with  $(x_i, x_i)_\sigma \neq 0$ . By Lemma 3.4 we choose  $\sigma_i \in U_V(D)$  such that  $\mathcal{R}_{\sigma_i} = \langle x_i \rangle$ . By definition,  $\sigma_1$  is a symmetry, so the result is proved by induction on the dimension of  $\mathcal{R}_\sigma$ .

(2) Suppose that  $\langle, \rangle|_{\mathcal{R}_\sigma}$  is alternating and not totally isotropic. Let us show it is impossible. Otherwise there are two different  $x, y \in \mathcal{R}_\sigma$ , such that  $\langle x, y \rangle \neq 0$ . Clearly the restriction of  $\langle, \rangle$  on the plane  $W$  generated by  $x, y$  is still non-degenerate. And we have  $\langle a, b \rangle = \epsilon \langle b, a \rangle$ , for  $a, b \in W$ , which implies that  $\langle, \rangle$  is symplectic contradicting to the hypothesis.

(3) Suppose now that  $\langle, \rangle|_{\mathcal{R}_\sigma}$  is alternating and totally isotropic. By assumption, there exists an anisotropic element  $x_1$  outside  $\mathcal{R}_\sigma$ , and moreover we may and do assume  $x_1 \in \mathcal{R}_\sigma^\perp$ . Choosing  $\sigma_1 \in U_V(D)$  with residual space  $\langle x_1 \rangle$ , we can assert  $\mathcal{R}_{\sigma \circ \sigma_1} = \mathcal{R}_\sigma \perp \langle x_1 \rangle$ . Applying above result of (1) to  $\sigma \circ \sigma_1$  gives the result.  $\square$

**3.1. The Eichler Transformation.** Let  $u, v$  be two vectors in  $V$  with  $\langle u, u \rangle = 0 = \langle u, v \rangle$ . For any  $d$  in the coset  $\frac{1}{2}\langle v, v \rangle + \mathcal{S}_D$  of  $D/\mathcal{S}_D$ , we define the so-called *Eichler transformation* related to  $u, v, d$  as follows(cf. [4, Page 214]):

$$e_{u,v,d}(x) = x + \epsilon u \langle v, x \rangle - (v + \epsilon u d) \langle u, x \rangle;$$

for instance,

$$\begin{cases} e_{u,v,0}(x) = x + \epsilon u \langle v, x \rangle - v \langle u, x \rangle, & \text{if } \langle v, v \rangle = 0, \\ e_{u,0,d}(x) = x - \epsilon u d \langle u, x \rangle, & \text{for } d \in \mathcal{S}_D. \end{cases}$$

It can be checked that (1)  $e_{u,v,d} \in U_V(D)^2$ ; (2)  $e_{u,0,d} = \tau_{u,\epsilon d}$ ; (3)  $e_{u,v,d} = e_{u,v,0} \circ e_{u,0,d}$  if  $\langle v, v \rangle = 0$ .<sup>3</sup> Let  $EU_V(D)$  be the group generated by all above Eichler transformations of  $U_V(D)$ .

**Remark 3.8.** Notice that when  $V$  is a hyperbolic space, the definition is compatible with that of Section 2.2.

*Proof.* We fix a hyperbolic basis  $\mathcal{A} = \{x_1, \dots, x_n; x_1^*, \dots, x_n^*\}$  of  $V$  so that  $\langle x_i, x_j \rangle = 0 = \langle x_i^*, x_j^* \rangle$ , and  $\langle x_i, x_j^* \rangle = \delta_{ij}$ . It can be easily checked that under above basis the elements  $e_{x_j, x_i^*, 0}$  ( $i \neq j$ ),  $e_{x_i, -x_j, 0}$  and  $e_{x_i^*, x_j^*, 0}$  correspond to the defined elements  $A_{ij}(r)$ ,  $B_{ij}(s)$  and  $C_{ij}(t)$  in Section 2.2 respectively. It remains to show that an arbitrary Eichler transformation  $e_{u,v,d}$  belongs to the used group  $EU_V(D)$  in Section 2.2. By Theorem 2.14, the group  $EU_V(D)$  does not depend on the choice of the hyperbolic basis of  $V$ , the case  $u = 0$  being trivial so we may assume  $u = x_1$ ,  $v = x_1 a_1 + \sum_{i=2}^n (x_i a_i + x_i^* a_i^*)$  and  $d = \frac{1}{2} \sum_{i=2}^n (\overline{a_i} a_i^* + \epsilon \overline{a_i^*} a_i) + d_1$ , for some  $d_1 = s_0 - \epsilon \overline{s_0} \in \mathcal{S}_D$ . Then there exists the equality:

$$e_{u,v,d} = e_{u,v_1,d_1} \circ \cdots \circ e_{u,v_n,d_n},$$

where  $v_1 = x_1 a_1$ ,  $d_1 = s_0 - \epsilon \overline{s_0}$ , and  $v_i = x_i a_i + x_i^* a_i^*$ ,  $d_i = \frac{1}{2}(\overline{a_i} a_i^* + \epsilon \overline{a_i^*} a_i)$  for  $i = 2, \dots, n$ . Notice that (1)  $e_{u,v_1,d_1} = e_{u,v_1,0} \circ e_{u,0,d_1}$ , and (2)  $e_{u,v_i,d_i} = (e_{u,x_i a_i^*/2,0} \circ e_{u,x_i a_i/2,0}) \circ (e_{u,x_i a_i/2,0} \circ e_{u,x_i^* a_i^*/2,0})$  for

<sup>2</sup> $\langle e_{u,v,d}(x), e_{u,v,d}(y) \rangle = \langle x + \epsilon u \langle v, x \rangle - (v + \epsilon u d) \langle u, x \rangle, y + \epsilon u \langle v, y \rangle - (v + \epsilon u d) \langle u, y \rangle \rangle = \langle x, y \rangle + \langle x, \epsilon u \langle v, y \rangle - \langle x, v \rangle \langle u, y \rangle - \langle x, \epsilon u d \langle u, y \rangle + \epsilon \langle v, x \rangle \langle u, y \rangle - \langle u, x \rangle \langle v, y \rangle + \langle u, x \rangle \langle v, v \rangle \langle u, y \rangle - \epsilon \langle u, x \rangle d \langle u, y \rangle \rangle = \langle x, y \rangle$  because  $d = \frac{1}{2}\langle v, v \rangle + x$  for some  $x \in V$  satisfying  $\overline{x} + \epsilon x = 0$ .

<sup>3</sup> $e_{u,0,d}(x) = x - \epsilon u d \langle u, x \rangle$ ;  $e_{u,v,0}(x - \epsilon u d \langle u, x \rangle) = x - \epsilon u d \langle u, x \rangle + \epsilon u \langle v, x - \epsilon u d \langle u, x \rangle \rangle - v \langle u, x - \epsilon u d \langle u, x \rangle \rangle = x - \epsilon u d \langle u, x \rangle + \epsilon u \langle v, x \rangle - v \langle u, x \rangle$ .

$i = 2, \dots, n$ ,<sup>4</sup> the Eichler transformations of part (2) and  $e_{u,v_1,0}$  all are the discussed elements of Section 2.2. The remaining  $e_{u,0,d_1} = \tau_{x_1, \epsilon d_1}$ , under the basis  $\mathcal{A}$ , corresponds to the matrix  $B_{11}(\overline{s_0})$  as well. This completes the proof.  $\square$

**Theorem 3.9** ([4, Page 335]).  $[\text{EU}_V(D), \text{EU}_V(D)] = \text{EU}_V(D)$ .

*Proof.* Let  $e_{u,v,d}$  be an arbitrary Eichler transformation in  $\text{EU}_V(D)$ . The isotropic vector  $u$  can be embedded in a hyperbolic plane  $H_1$ <sup>5</sup> of  $V$ . Suppose  $V = H_1 \perp W_1$ , and write  $v = x + y$  with  $x \in H_1$ ,  $y \in W_1$ . Suppose  $\langle y, y \rangle = a$ ; we can find  $z \in H_1$  with  $\langle z, z \rangle = -a$ <sup>6</sup>. Then  $y - z$  is an isotropic element, which can be embedded in a hyperbolic plane. As is a result that both  $u, v$  are in a hyperbolic space, denoted by  $V_1$ . So  $e_{u,v,d} \in [\text{EU}_{V_1}(D), \text{EU}_{V_1}(D)]$  by Corollary 2.4, Proposition 2.8, and Lemma 2.9, and the result follows.  $\square$

### 3.2. Typical decomposition of $\text{U}_V(D)$ .

**Lemma 3.10** ([4, Page 332]). *The group  $\text{EU}_V(D)$  acts transitively on the set of one-dimensional isotropic lines of  $V$ .*

*Proof.* Let  $u, v$  be two different isotropic vectors in  $V$ .

1) In case that  $u, v$  generate a hyperbolic plane, and  $\langle u, v \rangle = x \neq 0$ . For simplicity, we assume  $x = 1$ . If  $\mathcal{S}_D \neq 0$ , we take  $0 \neq a \in \mathcal{S}_D$ . Then by [4, Page 214, 5.2.9], the Eichler transformation  $e_{u,0,-\epsilon a^{-1}} \circ e_{v,0,a}$  maps  $(-ua^{-1})$  to  $v$  giving the result. If  $\mathcal{S}_D = 0$ , i.e.  $V$  being an orthogonal vector space over  $F$ , we can find an anisotropic element  $w \in \langle u, v \rangle^\perp$  with  $\frac{1}{2}\langle w, w \rangle = a$ . According to [4, Page 215], the Eichler transformation  $e_{u,wa^{-1},a^{-1}} \circ e_{v,w,a}$  maps  $(-ua^{-1})$  to  $v$ .<sup>7</sup>

2) In case  $\langle u \rangle \perp \langle v \rangle$ , by the fact that  $\dim(\langle u \rangle^\perp \cup \langle v \rangle^\perp) = n - 1$ , we can choose an element  $w \in V$  neither orthogonal to  $u$  nor to  $v$ , and the result follows from the first case.  $\square$

**Lemma 3.11** ([4, Page 333]). *The group  $\text{EU}_V(D)$  acts transitively on the set of hyperbolic planes of  $V$ .*

*Proof.* We choose two hyperbolic planes  $H_1 = \langle x, y \rangle$ ,  $H_2 = \langle u, v \rangle$ , with  $x, y, u$ , and  $v$  all being isotropic vectors. By Lemma 3.10, we can find  $\sigma \in \text{EU}_V(D)$  such that  $\sigma(x) = u$  so we may assume  $H_1 = \langle x, y \rangle$  and  $H_2 = \langle x, v \rangle$ , with  $\langle x, y \rangle = 1 = \langle x, v \rangle$ . Set  $\langle y, v \rangle = a$ . By [4, Page 331, 6.3.7], we have  $e_{x,v-y,-a}(v) = y$ .<sup>8</sup>  $\square$

**Theorem 3.12** ([4, Page 333]).  $\text{U}_V(D) = \text{U}_H(D) \cdot \text{EU}_V(D)$ .

*Proof.* By  $\sigma(e_{u,v,d})\sigma^{-1} = e_{\sigma(u),\sigma(v),d}$  for  $\sigma \in \text{U}_V(D)$ , we know  $\text{EU}_V(D)$  is a normal subgroup of  $\text{U}_V(D)$ . By Proposition 3.6, 3.7, it suffices to show that every one-dimensional transformation is contained on the right-hand side of the equality. Let  $\sigma$  be the one with residual line  $\mathcal{R}_\sigma$ . So  $\mathcal{R}_\sigma$  is contained in a

<sup>4</sup> It arises from (1)  $e_{u,x_1 a_i/2,0} \circ e_{u,x_1^* a_i^*/2,0}(x) = x + \epsilon u \langle \frac{v_i}{2}, x \rangle - (\frac{v_i}{2} + \epsilon u \frac{\overline{a_i} \overline{a_i^*}}{4}) \langle u, x \rangle$ , and (2)  $e_{u,x_1^* a_i^*/2,0} \circ e_{u,x_1 a_i/2,0}(x) = x + \epsilon u \langle \frac{v_i}{2}, x \rangle - (\frac{v_i}{2} + u \frac{\overline{a_i} \overline{a_i^*}}{4}) \langle u, x \rangle$ , for  $x \in V$ .

<sup>5</sup> We choose a  $w \in V$  such that  $\langle u, w \rangle = 1 \neq 0$ . If  $\langle w, w \rangle = 0$ , then  $\{u, w\}$  is a hyperbolic basis. If  $\langle w, w \rangle = \overline{a} \neq 0$ , then  $\overline{a} = \epsilon a$  and  $\{-u \cdot \frac{\overline{a}}{2}, u - w \cdot \frac{2}{a}\}$  is a hyperbolic basis in  $V$ .

<sup>6</sup> Let  $\{e_1, e_2\}$  be a hyperbolic basis of  $H_1$  so that  $\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = 0$ ,  $\langle e_1, e_2 \rangle = 1$ . Then the element  $z = e_1 - e_2 \frac{a}{2}$  satisfies the condition.

<sup>7</sup> Let us check both cases separately. (1)  $e_{v,0,a}(-ua^{-1}) = -ua^{-1} - \epsilon va \langle v, -ua^{-1} \rangle = -ua^{-1} + v$ , and  $e_{u,0,-\epsilon a^{-1}}(-ua^{-1} + v) = -ua^{-1} + v - \epsilon u \langle -\epsilon a^{-1}, -ua^{-1} + v \rangle = v$ ; (2)  $e_{v,w,a}(-ua^{-1}) = -ua^{-1} + v \langle w, -ua^{-1} \rangle - (w + va) \langle v, -ua^{-1} \rangle = -ua^{-1} + wa^{-1} + v$ , and  $e_{u,wa^{-1},a^{-1}}(-ua^{-1} + wa^{-1} + v) = -ua^{-1} + wa^{-1} + v + u \langle wa^{-1}, wa^{-1} \rangle - (wa^{-1} + ua^{-1}) = v$ .

<sup>8</sup>  $\langle x, x \rangle = 0 = \langle x, v - y \rangle$ ,  $\frac{1}{2}\langle v - y, v - y \rangle + a = \frac{1}{2}(-a - \epsilon \overline{a}) + a = \frac{1}{2}(a - \epsilon \overline{a}) \in \mathcal{S}_D$ , and  $e_{x,v-y,-a}(v) = v + x \langle (v - y)\epsilon, v \rangle - (v - y) \langle x, v \rangle + x \epsilon a \langle x, v \rangle = v - x a \epsilon - (-y + v) + x \epsilon a = y$ .

hyperbolic plane  $H_1$ <sup>9</sup>. By Lemma 3.11, taking  $\sigma_1 \in \text{EU}_V(D)$ , such that  $\sigma_1(H_1) = H$ , we then have  $\mathcal{R}_{\sigma_1\sigma\sigma_1^{-1}} \subseteq H$ , and  $\sigma_1\sigma\sigma_1^{-1} \in \text{U}_H(D)$  stabilizing  $H^\perp$ .  $\square$

As a consequence we obtain

**Proposition 3.13.** *There is a canonical surjective map  $\text{U}(H)/[\text{U}(H), \text{U}(H)] \longrightarrow \text{U}_V(D)/[\text{U}_V(D), \text{U}_V(D)]$ .*

#### 4. CLASSIFICATION OF THE ANISOTROPIC $\epsilon$ -HERMITIAN SPACES OVER $D$

In this section, let us present the classification of the anisotropic  $\epsilon$ -hermitian spaces over  $D$  in some cases.

**4.1. Hermitian space over  $D$ .** Let us rephrase one result from [11, Page 7].

**Theorem 4.1.** *Up to isometry,*

- *an anisotropic quadratic vector space over  $F$  has one of the following forms:*
  - (i)  $F(a)$ , for  $a \in F^\times$  modulo  $(F^\times)^2$ , with the canonical form  $x \mapsto ax^2$ ,  $x \in F$ ;
  - (ii)  $F_1(a)$ , for any quadratic field extension  $F_1$  of  $F$ ,  $a \in F^\times$  modulo  $N_{F_1/F}(F_1^\times)$  with the form  $x \mapsto aN_{F_1/F}(x)$ ,  $x \in F_1$ ;
  - (iii)  $\mathbb{H}^0(a)$ , for  $a \in F^\times$  modulo  $(F^\times)^2$ , with the form  $x \mapsto \tau(\mathbb{x})a\mathbb{x}$ ,  $\mathbb{x} \in \mathbb{H}^0$ ;
  - (iv)  $\mathbb{H}$ , with the form  $\mathbb{x} \mapsto \text{Nrd}(\mathbb{x})$ ,  $\mathbb{x} \in \mathbb{H}$ .
- *an anisotropic hermitian vector space over  $E$  has one of the following forms:*
  - (i)  $E(a)$ , for  $a \in F^\times$  modulo  $N_{E/F}(E^\times)$ , with the form  $(x, y) \mapsto a\tau(x)y$ ,  $x, y \in E$ ;
  - (ii)  $\mathbb{H}$ , with the form  $(\mathbb{x}, \mathbb{y}) \mapsto \text{Tr}_{\mathbb{H}/E}(\tau(\mathbb{x})\mathbb{y})$ ,  $\mathbb{x}, \mathbb{y} \in \mathbb{H}$  (see Section 6.1 for details);
- *an anisotropic right hermitian vector space over  $\mathbb{H}$  has the following form:  $\mathbb{H}$ , with the form  $(\mathbb{x}, \mathbb{y}) \mapsto \tau(\mathbb{x})\mathbb{y}$ ,  $\mathbb{x}, \mathbb{y} \in \mathbb{H}$ .*

**4.2. Skew Hermitian spaces over  $E$ .** We now let  $(V, \langle, \rangle)$  be an anisotropic hermitian space over  $E$ . By Hilbert's Theorem 90, let us take an element  $\mu$  of  $E^\times$  such that  $\bar{\mu}/\mu = -1$ . As is known that multiplicity of  $\langle, \rangle$  by  $\mu$  will give a skew hermitian form  $\mu\langle, \rangle$  on  $V$ . As studied above, we have

**Proposition 4.2.** *Up to isometry, an anisotropic skew space over  $E$  has one of the following forms:*

- (i)  $E(a)$ , for  $a \in F^\times$  modulo  $N_{E/F}(E^\times)$ , with the form  $(x, y) \mapsto a\mu\tau(x)y$ ,  $x, y \in E$ ;
- (ii)  $\mathbb{H}$ , with the form being given as  $(\mathbb{x}, \mathbb{y}) \mapsto \mu \text{Tr}_{\mathbb{H}/E}(\tau(\mathbb{x})\mathbb{y})$ ,  $\mathbb{x}, \mathbb{y} \in \mathbb{H}$ .

**4.3. Tsukamoto's classification.** Now let  $V$  be a nondegenerate right (resp. left) skew hermitian space over the quaternion algebra  $\mathbb{H}$  of dimension  $n$ ; it has a Witt decomposition  $V = \bigoplus_{i=1}^n \mathbb{H}(\mathfrak{a}_i)$  (resp.  $(\mathfrak{a}_i)\mathbb{H}$ ) for some  $0 \neq \mathfrak{a}_i \in \mathbb{H}^0$  with  $\mathfrak{a}_i^2 = \alpha_i$ . The discriminant of  $V$  is defined by  $(\alpha_1) \cdots (\alpha_n)(F^\times)^2 \in F^\times/(F^\times)^2$ .

**Theorem 4.3** (Tsukamoto). *Let  $(V, \langle, \rangle)$  be an anisotropic skew hermitian space over  $\mathbb{H}$ .*

- (1) *Up to isometry,  $V$  is uniquely determined by its dimension and by its discriminant.*
- (2)  *$V$  has the possible dimension 1, 2, or 3.*
- (3) *When  $\dim V = 1, 2$ , the discriminant of  $V$  can be any  $\alpha(F^\times)^2$ , except  $(F^\times)^2$ .*

<sup>9</sup> According to the proof of Theorem 3.9, it reduces to assume that  $\mathcal{R}_\sigma = \langle x \rangle$  with  $\langle x, x \rangle = a \neq 0$ . Obviously  $\langle, \rangle$  is not a symplectic form. Suppose  $x = y + z$  with  $y \in H^\perp, z \in H$ . If  $y = 0$ ,  $x \in H$ , we are done. If  $z = 0$ ,  $x \in H^\perp$ , we fix a hyperbolic basis  $\{e_1, e_2\}$  of  $H$  so that  $\langle e_1, e_1 \rangle = 0 = \langle e_2, e_2 \rangle$  and  $\langle e_1, e_2 \rangle = 1$ ; then  $\langle e_1 - e_2 \cdot \frac{a}{8}, e_1 - e_2 \cdot \frac{a}{8} \rangle = -\frac{a}{8}\epsilon - \frac{a}{8} = -\frac{a}{4}$ . Now the elements  $\frac{3}{2} \pm (e_1 - e_2 \cdot \frac{a}{8})$  both are isotropic vectors, and they span a hyperbolic plane containing  $x$ . If  $y \neq 0, z \neq 0$ , we assume  $y$  belongs to a hyperbolic plane  $H_1$  so that  $x$  is contained in a hyperbolic space  $W$  of 4 dimension. Choose a hyperbolic basis  $\{e_1, e_2, e_1^*, e_2^*\}$  of  $W$ , and write  $x = e_1a_1 + e_2a_2 + e_1^*a_1^* + e_2^*a_2^*$ . Then the vector  $\text{Span}\{e_1a_1 + e_2a_2, e_1^*a_1^* + e_2^*a_2^*\}$  is a hyperbolic plane, and contains  $x$

(4) When  $\dim V = 3$ , the discriminant of  $V$  can only be  $(F^\times)^2$ .

*Proof.* See [22, Theorem 3]. □

Now let us understand Tsukamoto's result by the aid of the structure of  $\mathbb{H}$ . For the quaternion algebra  $\mathbb{H}$  over  $F$ , it is known (e.g. [1, Chapter 13]) that  $k_{\mathbb{H}} = \mathfrak{D}/\mathfrak{P}$  is a field extension of  $k_F$  (the residue field of  $F$ ) of degree 2. We choose an element  $\varpi$  of  $\mathbb{H}^\times$  such that  $\text{Nrd}(\varpi)$  is a prime element of  $F$ . Clearly  $\varpi \notin (F^\times)^2$ , so we assume that  $\varpi$  is of pure quaternion. Let  $0 \neq \xi$  be another nontrivial element of pure quaternion such that  $\varpi\xi = -\xi\varpi$  and  $\{1, \varpi, \xi, \varpi\xi\}$  forms a standard basis of  $\mathbb{H}$ . We assume that  $F(\xi)/F$  is a unramified quadratic field extension. We fix an element  $e_{-1} \in F(\xi)$  such that  $\text{Nrd}(e_{-1}) = -1$ <sup>10</sup>.

**Corollary 4.5.** *Let  $(V, \langle, \rangle)$  be a right anisotropic skew hermitian space over  $\mathbb{H}$ . Up to isometry,  $V$  has one of the following forms:*

- (1)  $\mathbb{H}(\mathfrak{i})$ , for  $\mathfrak{i} = \xi, \varpi, \xi\varpi$ .
- (2)  $\mathbb{H}(\mathfrak{i}) \oplus \mathbb{H}(\mathfrak{j})$ , for  $(\mathfrak{i}, \mathfrak{j}) = (\xi, e_{-1}\varpi), (\xi, e_{-1}\xi\varpi)$  or  $(\varpi, e_{-1}\xi\varpi)$ .
- (3)  $\mathbb{H}(\xi) \oplus \mathbb{H}(\varpi) \oplus \mathbb{H}(e_{-1}\xi\varpi)$ .

## 5. THE ANISOTROPIC UNITARY GROUPS I.

In this section we shall study the mostly simple anisotropic unitary group over the quaternion algebra  $\mathbb{H}$ . Before presenting the results, let us cite some results from Carl Riehm's paper [17] involving those subgroups of the norm one group of  $\mathbb{H}^\times$ .

Recall that for the quaternion algebra  $\mathbb{H}$  over  $F$ ,  $\mathfrak{D} = \{\mathfrak{d} \in \mathbb{H} \mid |\text{Nrd}(\mathfrak{d})|_F \leq 1\}$ ,  $\mathfrak{P} = \{\mathfrak{d} \in \mathbb{H} \mid |\text{Nrd}(\mathfrak{d})|_F < 1\}$ ,  $U_{\mathbb{H}} = \{\mathfrak{d} \in \mathbb{H} \mid |\text{Nrd}(\mathfrak{d})|_F = 1\}$ ,  $\mathbb{H}^0 = \{\mathfrak{d} \in \mathbb{H} \mid \text{Trd}(\mathfrak{d}) = 0\}$ ,  $\text{SL}_1(\mathbb{H}) = \{\mathfrak{d} \in \mathbb{H}^\times \mid \text{Nrd}(\mathfrak{d}) = 1\}$ . Let  $\{1, \xi, \varpi, \xi\varpi\}$  be a standard basis given in Section 4.3. The ring of integers of a local field  $K$  will be denoted by  $\mathfrak{O}_K$ , its unique maximal ideal by  $\mathfrak{p}_K$ , its residue field by  $k_K$ , its group of units by  $U_K$ .

**Lemma 5.1.**  $[\mathbb{H}^\times, \mathbb{H}^\times] = \text{SL}_1(\mathbb{H})$ .

*Proof.* It is clear that  $[\mathbb{H}^\times, \mathbb{H}^\times] \subseteq \text{SL}_1(\mathbb{H})$  and  $-1 \in \text{SL}_1(\mathbb{H})$ . Let  $a + \mathfrak{i} \in \text{SL}_1(\mathbb{H})$ , such that  $a \in F$  and  $\mathfrak{i} \neq 0$  is an element of pure quaternion. Then we have  $\text{Nrd}(a + \mathfrak{i}) = a^2 - \mathfrak{i}^2 = 1$ . We choose  $\mathfrak{j} \in \mathbb{H}^\times$ , such that  $\mathfrak{i}\mathfrak{j} = -\mathfrak{j}\mathfrak{i} = \mathfrak{k}$  and  $\{1, \mathfrak{i}, \mathfrak{j}, \mathfrak{k}\}$  forms a standard basis of  $\mathbb{H}$ . Let us consider the following equation:  $(a + \mathfrak{i})(x + \mathfrak{i}y) = x - \mathfrak{i}y$ , i.e.  $\begin{cases} (a-1)x + \mathfrak{i}^2y = 0 \\ x + (a+1)y = 0 \end{cases}$ ; the determinant of its coefficient matrix is just  $\det \begin{pmatrix} (a-1) & \mathfrak{i}^2 \\ 1 & (a+1) \end{pmatrix} = a^2 - 1 - \mathfrak{i}^2 = 0$ . So it has a non-zero solution  $x_0 + \mathfrak{i}y_0$  for  $x_0, y_0 \in F$ . By calculation, we see  $a + \mathfrak{i} = \mathfrak{j}(x_0 + \mathfrak{i}y_0)\mathfrak{j}^{-1}(x_0 + \mathfrak{i}y_0)^{-1} \in [\mathbb{H}^\times, \mathbb{H}^\times]$ . □

**5.1. Riehm's results.** The following two lemmas are coming from Riehm's paper [17].

- Lemma 5.2.** (1)  $(\text{SL}_1(\mathbb{H}) \cap (1 + \mathfrak{P})) / (\text{SL}_1(\mathbb{H}) \cap (1 + \mathfrak{P}^2))$  is isomorphic to  $\mathfrak{D}/\mathfrak{P}$ .  
 (2)  $(\text{SL}_1(\mathbb{H}) \cap (1 + \mathfrak{P})) / (\text{SL}_1(\mathbb{H}) \cap (1 + \mathfrak{P}^2))$  is a simple  $\text{SL}_1(\mathbb{H})$ -module. (The action is induced by inner automorphism of  $\text{SL}_1(\mathbb{H})$  on itself.)

**Lemma 4.4** ([10, Page 55, Proposition 2.9]). *Let  $F(\xi)$  be a unramified quadratic field extension of  $F$ . Then  $-1$  is a norm from  $F(\xi)$  to  $F$ .*



*Proof.* We follow the proof of [17]. (1): Consider the operator  $1 + \mathfrak{P}/1 + \mathfrak{P}^2 \longrightarrow \mathfrak{D}/\mathfrak{P}; 1 + \varpi a \longmapsto a \bmod \mathfrak{P}$ . Its restriction to  $\mathbb{S}\mathbb{L}_1(\mathbb{H})$  yields an embedding  $(\mathbb{S}\mathbb{L}_1(\mathbb{H}) \cap (1 + \mathfrak{P})) / (\mathbb{S}\mathbb{L}_1(\mathbb{H}) \cap (1 + \mathfrak{P}^2)) \hookrightarrow \mathfrak{D}/\mathfrak{P}$ . Since  $\mathfrak{D}/\mathfrak{P}$  is isomorphic to  $\mathfrak{O}_{F(\xi)}/\mathfrak{p}_{F(\xi)}$ , we have  $1 + \mathfrak{P} = 1 + \mathfrak{O}_{F(\xi)}/\mathfrak{p}_{F(\xi)}\varpi$ . For any  $\alpha \in 1 + \mathfrak{P}$ , say  $\alpha = 1 + \varpi a$  with  $a \in \mathfrak{O}_{F(\xi)}$ , its reduced norm  $\text{Nrd}(\alpha)$  is just  $1 - \varpi^2 \text{N}_{F(\xi)/F}(a) \in 1 + \mathfrak{p}_F$ . As  $F(\xi)/F$  is a unramified field extension, it is known that there exists  $\beta \in 1 + \mathfrak{p}_{F(\xi)}$  such that  $\text{N}_{F(\xi)/F}(\beta) = 1 - \varpi^2 \text{N}_{F(\xi)/F}(a) = \text{Nrd}(\alpha)$ . Now the element  $g = \alpha\beta^{-1}$  belongs to  $\mathbb{S}\mathbb{L}_1(\mathbb{H})$  and satisfies  $g \equiv \alpha \bmod \mathfrak{P}^2$  as required. This proves the first statement. (2): Let  $T = \{x \in \mathfrak{O}_{F(\xi)} \mid x^{q^2} = x\}$  be the set of the Teichmüller representatives of  $k_{F(\xi)}$  in  $\mathfrak{O}_{F(\xi)}$ . By uniqueness,  $T$  is  $\text{Gal}(F(\xi)/F)$ -stable, and any  $t \in T$  whose norm is  $\equiv 1 \bmod \mathfrak{p}_F$  should belong to  $\mathbb{S}\mathbb{L}_1(\mathbb{H})$ . Now such  $t$  acts on  $g \in \mathbb{S}\mathbb{L}_1(\mathbb{H}) \cap (1 + \mathfrak{P})$ , say  $g \equiv 1 + \varpi a \bmod \mathfrak{P}^2$  with  $a \in \mathfrak{O}_{F(\xi)}$ , as

$$tgt^{-1} \equiv 1 + \varpi a(\bar{t}/t) \bmod \mathfrak{P}^2.$$

Under the isomorphism  $(\mathbb{S}\mathbb{L}_1(\mathbb{H}) \cap (1 + \mathfrak{P})) / (\mathbb{S}\mathbb{L}_1(\mathbb{H}) \cap (1 + \mathfrak{P}^2)) \simeq \mathfrak{O}_{F(\xi)}/\mathfrak{p}_{F(\xi)} (\simeq k_{F(\xi)})$ , the above action of  $t$  is transferred as  $a_0 \longmapsto a_0 t_0^{q-1}$  for  $a_0 \in k_{F(\xi)}$ ,  $t_0$  the projection of  $t$  in  $k_{F(\xi)}$ . Note that  $t_0$  can be any element in  $S = \{s \in k_{F(\xi)} \mid s^{q+1} = 1\}$  so that we can choose  $t_0 \notin F_q$ . Hence  $k_{F(\xi)}$  of cardinality  $q^2$  should be a simple  $\mathbb{S}\mathbb{L}_1(\mathbb{H})$ -module.  $\square$

**Lemma 5.3.** (1)  $[\mathbb{S}\mathbb{L}_1(\mathbb{H}), \mathbb{S}\mathbb{L}_1(\mathbb{H})] = \mathbb{S}\mathbb{L}_1(\mathbb{H}) \cap (1 + \mathfrak{P})$ .

(2)  $\mathbb{S}\mathbb{L}_1(\mathbb{H})/[\mathbb{S}\mathbb{L}_1(\mathbb{H}), \mathbb{S}\mathbb{L}_1(\mathbb{H})]$  is isomorphic with the subgroup  $\mathbb{S}\mathbb{L}_1(k_{\mathbb{H}})$  of elements of  $k_{\mathbb{H}}$  with norms 1 in  $k_F$ .

*Proof.* We follow the proof of [17]. Let  $x = a + \varpi b, y = c + \varpi d \in \mathbb{S}\mathbb{L}_1(\mathbb{H}) \subseteq \mathbb{U}_{\mathbb{H}}$  for  $a, b, c, d \in \mathfrak{O}_{F(\xi)}$ . It follows that  $xy - yx \equiv 0 \bmod \mathfrak{P}$ . So  $[x, y] = 1 + (xy - yx)x^{-1}y^{-1}$  belongs to  $1 + \mathfrak{P}$ . Now we show the reverse. Suppose that  $g_1 = 1 + \varpi a \in (\mathbb{S}\mathbb{L}_1(\mathbb{H}) \cap (1 + \mathfrak{P}))$  with  $a \notin \mathfrak{P}$ , and  $g_2 = b_2 \in \mathbb{S}\mathbb{L}_1(\mathbb{H}) \cap \mathfrak{O}_{F(\xi)}$  ( $\in \mathbb{U}_{F(\xi)}$ ). Then by Lemma 5.2(1) and  $\mathfrak{D}/\mathfrak{P} \simeq \mathfrak{O}_{F(\xi)}/\mathfrak{p}_{F(\xi)}$ , we assume  $g_1 = 1 + \varpi a_1 \bmod \mathfrak{P}^2$  for some  $a_1 \in \mathbb{U}_{F(\xi)}$ . Now

$$[g_1, g_2] \equiv 1 + (g_1 g_2 - g_2 g_1) g_1^{-1} g_2^{-1} \equiv 1 + \varpi a_1 \overline{b_2} (b_2 - \overline{b_2}) \bmod \mathfrak{P}^2.$$

Since the set  $T$  of the Teichmüller representatives of  $k_{F(\xi)}$  in  $\mathfrak{O}_{F(\xi)}$  contains at least  $q + 1$  elements of norm 1, we can choose certain  $b_2$  such that  $\overline{b_2} \neq b_2$ , and then  $[g_1, g_2] \notin 1 + \mathfrak{P}^2$ . By Lemma 5.2(2), the  $\mathbb{S}\mathbb{L}_1(\mathbb{H})$ -module  $[\mathbb{S}\mathbb{L}_1(\mathbb{H}), \mathbb{S}\mathbb{L}_1(\mathbb{H})]$  coincides with  $\mathbb{S}\mathbb{L}_1(\mathbb{H}) \cap (1 + \mathfrak{P})$ . Now consider the natural projection  $p : \mathbb{U}_{\mathbb{H}} \longrightarrow \mathbb{U}_{\mathbb{H}}/(1 + \mathfrak{P})$  restricted to  $\mathbb{S}\mathbb{L}_1(\mathbb{H})$ . Clearly its image belongs to  $\mathbb{S}\mathbb{L}_1(k_{\mathbb{H}})$ . Conversely the set  $\mathbb{S}\mathbb{L}_1(\mathbb{H}) \cap T$  has distinct  $q + 1$  images in  $k_{\mathbb{H}}$ , so the second statement holds.  $\square$

**Remark 5.4.** Suppose now that  $\mathbb{H}$  is merely a division algebra over its centre  $F$  of dimension  $d^2$ . Notations being given at beginning, then the results of Lemma 5.3 also hold.

*Proof.* The proof is similar as above. See [17, Section 5 and Theorem 7(iii)(2)].  $\square$

**5.2. One-dimensional  $\epsilon$ -Hermitian space over  $\mathbb{H}$ .** Let  $\{1, \mathfrak{i}, \mathfrak{j}, \mathfrak{k}\}$  be a standard base of  $\mathbb{H}$  such that  $\mathfrak{i} \cdot \mathfrak{j} = -\mathfrak{j} \cdot \mathfrak{i} = \mathfrak{k}$ , and  $\mathfrak{i}^2 = -\alpha, \mathfrak{j}^2 = -\beta$ . Set  $F_1 = F(\mathfrak{i})$ , and  $F_2 = F(\mathfrak{j})$ .

**Lemma 5.5.** Let  $(V = \mathbb{H}(\mathfrak{i}), \langle, \rangle)$  be a right skew hermitian space over  $\mathbb{H}$  of dimension 1. Then  $\text{U}(V) = \mathbb{S}\mathbb{L}_1(F_1) = \{x \in F_1^\times \mid \text{N}_{F_1/F}(x) = 1\}$  and  $\text{GU}(V) = \langle F_1^\times, \mathfrak{j} \rangle = F_1^\times \cup F_1^\times \mathfrak{j}$ .

*Proof.* An element  $\alpha_0 + \alpha_1 \mathfrak{j} \in \text{U}(V)$  with  $\alpha_0, \alpha_1 \in F_1$ , by definition, satisfies  $(\overline{\alpha_0} - \alpha_1 \mathfrak{j}) \mathfrak{i} (\alpha_0 + \alpha_1 \mathfrak{j}) = \mathfrak{i}$ ; this means  $\alpha_1 = 0$  and  $\text{N}_{F_1/F}(\alpha_0) = 1$ , or  $\alpha_0 = 0$  and  $\text{N}_{F_1/F}(\alpha_1) \mathfrak{j}^2 = 1$ . But the second case contradicts to  $\mathfrak{j}^2 \notin \text{N}_{F_1/F}(F_1^\times)$ , so  $\text{U}(V) = \mathbb{S}\mathbb{L}_1(F_1)$ . Similarly, if  $g = \alpha_0 + \alpha_1 \mathfrak{j} \in \text{GU}(V)$ , then

$(\overline{\alpha_0} - \alpha_1 \mathfrak{j}) \mathfrak{i} (\alpha_0 + \alpha_1 \mathfrak{j}) = \lambda(g) \mathfrak{i}$  for some  $\lambda(g) \in F^\times$ . By calculation, we see that  $\alpha_0 = 0$ , or  $\alpha_1 = 0$ , so the result follows.  $\square$

The following result is immediate:

**Lemma 5.6.** *Let  $(V = \mathbb{H}, \langle, \rangle = \text{Trd})$  be a right hermitian space over  $\mathbb{H}$  of dimension 1. Then  $U(V) = \mathbb{S}L_1(\mathbb{H})$  and  $GU(V) = \mathbb{H}^\times$ .*

## 6. THE ANISOTROPIC UNITARY GROUPS II.

In this section, we will denote by  $(V, \langle, \rangle)$  an anisotropic hermitian space over  $E$ , let  $U(V, \langle, \rangle)$  be the group of isometries of  $(V, \langle, \rangle)$ ,  $GU(V, \langle, \rangle)$  the group of isometries of similitudes of  $(V, \langle, \rangle)$ . Let  $\mathbb{S}L_1(E) = \{x \in E^\times \mid N_{E/F}(x) = 1\}$ .

**Proposition 6.1.** *If  $\dim_E(V) = 1$ , then  $U(V, \langle, \rangle) = \mathbb{S}L_1(E)$ , and  $GU(V, \langle, \rangle) = E^\times$ .*

*Proof.* Obviously.  $\square$

**6.1. The anisotropic hermitian space of 2 dimension.** Now let us discuss in more detail when  $\dim_E(V) = 2$ . In this case, by Theorem 4.1, we assume  $E = F(\mathfrak{i})$ , for some element  $\mathfrak{i}$  of pure quaternion with  $\mathfrak{i}^2 = -\alpha \in F^\times$  (cf. [1, Page 326, Proposition]). We will write  $\overline{\mathfrak{d}} = \tau(\mathfrak{d})$  for  $\mathfrak{d} \in \mathbb{H}$ . By [20, Page 358], we can choose an element  $\mathfrak{j}$  of  $\mathbb{H}$ , such that  $\mathfrak{i}\mathfrak{j} = -\mathfrak{j}\mathfrak{i} = \mathfrak{k}$ ,  $\mathfrak{j}^2 = -\beta \in F^\times$ , and  $\{1, \mathfrak{i}, \mathfrak{j}, \mathfrak{k}\}$  forms a standard basis of  $\mathbb{H}$ . There is a decomposition of  $E$ -vector space:

$$\mathbb{H} = E \oplus \mathfrak{j}E.$$

Let  $\text{Tr}_{\mathbb{H}/E}$  denote the canonical projection from  $\mathbb{H}$  to  $E$  defined by

$$\text{Tr}_{\mathbb{H}/E}(e_1 + \mathfrak{j}e_2) = e_1, \quad e_i \in E.$$

Now, we can define an  $E$ -hermitian form  $\langle, \rangle$  on  $\mathbb{H}$  as follows:

$$\langle e_1 + \mathfrak{j}e_2, e'_1 + \mathfrak{j}e'_2 \rangle = \text{Tr}_{\mathbb{H}/E}(\overline{(e_1 + \mathfrak{j}e_2)}(e'_1 + \mathfrak{j}e'_2)) = \overline{e_1}e'_1 + \overline{e_2}\mathfrak{j}e'_2 = \overline{e_1}e'_1 + \beta\overline{e_2}e'_2;$$

if given  $a, a' \in E$ , we then have

$$\langle (e_1 + \mathfrak{j}e_2)a, (e'_1 + \mathfrak{j}e'_2)a' \rangle = \overline{a}\overline{e_1}e'_1a' + \beta\overline{a}\overline{e_2}e'_2a' = \overline{a}\langle e_1 + \mathfrak{j}e_2, e'_1 + \mathfrak{j}e'_2 \rangle a'.$$

Moreover, if  $\langle e_1 + \mathfrak{j}e_2, e_1 + \mathfrak{j}e_2 \rangle = \text{Nrd}(e_1) + \beta\text{Nrd}(e_2) = 0$ , then  $\text{Nrd}(e_1) = \text{Nrd}(e_2) = 0$ , and consequently  $e_1 = e_2 = 0$ . So  $(\mathbb{H}, \langle, \rangle)$  is the unique anisotropic space over  $E$  of dimension 2, up to isometry.

**6.2. The group  $U(V, \langle, \rangle)$ .** For simplicity, we choose a basis  $\{1, \mathfrak{j}\}$  of  $\mathbb{H}$ . Under such basis, we identify  $U(V, \langle, \rangle)$  with the unitary matrix group  $U_V(D)$  consisting of elements  $G = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \in \text{GL}_2(E)$  such that

$$G^* \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} G = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}, \quad (6.1)$$

where  $*$  :  $\text{GL}_2(E) \longrightarrow \text{GL}_2(E)$  is the conjugate transpose operator. By calculation, (6.1) is equivalent to,

$$\begin{aligned} \text{N}_{E/F}(\alpha_{11}) + \beta\text{N}_{E/F}(\alpha_{21}) &= 1, \\ \text{N}_{E/F}(\alpha_{12}) + \beta\text{N}_{E/F}(\alpha_{22}) &= \beta, \\ \overline{\alpha_{11}}\alpha_{12} + \beta\overline{\alpha_{21}}\alpha_{22} &= 0. \end{aligned} \quad (6.2)$$



**Proposition 6.2.** (1) *There is a canonical embedding  $\mathbb{SL}_1(\mathbb{H}) \rightarrow U(V, \langle, \rangle); \alpha_1 + \mathfrak{j}\alpha_2 \mapsto \begin{pmatrix} \alpha_1 & -\beta\overline{\alpha_2} \\ \alpha_2 & \overline{\alpha_1} \end{pmatrix}$ .*

(2)  $U(V, \langle, \rangle)$  contains a subgroup  $\mathfrak{U} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \beta_1 \end{pmatrix} \mid \beta_1 \in \mathbb{SL}_1(E) \right\}$ . Moreover,  $U(V, \langle, \rangle) = \{H \cdot A \mid H \in \mathbb{SL}_1(\mathbb{H}), A \in \mathfrak{U}\}$ .

*Proof.* The first part of (2) follows from above equations (6.2). By definition, an element  $\alpha_1 + \mathfrak{j}\alpha_2 \in \mathbb{SL}_1(\mathbb{H})$  belongs to  $U(V, \langle, \rangle)$ , and sends 1 to  $\alpha_1 + \mathfrak{j}\alpha_2$ ,  $\mathfrak{j}$  to  $-\beta\overline{\alpha_2} + \mathfrak{j}\overline{\alpha_1}$ , which gives the result (1).

Further,  $G = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \in U(V, \langle, \rangle)$  can be written in the form:

- (1)  $G = \begin{pmatrix} \alpha_{11} & \alpha_{12}\overline{\alpha_{11}}\alpha_{22}^{-1} \\ \alpha_{21} & \overline{\alpha_{11}} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \overline{\alpha_{11}}^{-1}\alpha_{22} \end{pmatrix}$  with  $\alpha_{12}\overline{\alpha_{11}}\alpha_{22}^{-1} = -\beta\overline{\alpha_{21}}$  by equations (6.2), if  $\alpha_{11} \neq 0$ , and  $\alpha_{22} \neq 0$ .
- (2)  $G = \begin{pmatrix} 0 & -\beta\overline{\alpha_{21}} \\ \alpha_{21} & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \alpha_{12}(-\beta\overline{\alpha_{21}})^{-1} \end{pmatrix}$  with  $N_{E/F}(\alpha_{12}(-\beta\overline{\alpha_{21}})^{-1}) = 1$  by equations (6.2), if  $\alpha_{11} = \alpha_{22} = 0$  is possible.

□

**Proposition 6.3.**  $[U(V, \langle, \rangle), U(V, \langle, \rangle)] = [\mathbb{SL}_1(\mathbb{H}), \mathbb{SL}_1(\mathbb{H})] \simeq \mathbb{SL}_1(\mathbb{H}) \cap (1 + \mathfrak{P})$

*Proof.* We prove the result along the cases given in Corollary 4.5. In case  $\mathfrak{i} = \xi, \mathfrak{j} = \varpi$  or  $\xi\varpi$ , the commutator of two elements  $G_1 = \begin{pmatrix} \alpha_1 & -\beta\overline{\alpha_2} \\ \beta_1\alpha_2 & \beta_1\overline{\alpha_1} \end{pmatrix}$ , and  $G_2 = \begin{pmatrix} \alpha'_1 & -\beta\overline{\alpha'_2} \\ \beta'_1\alpha'_2 & \beta'_1\overline{\alpha'_1} \end{pmatrix}$  in  $U(V, \langle, \rangle)$  is presented by

$$[G_1, G_2] \equiv \left[ \begin{pmatrix} \alpha_1 & 0 \\ \beta_1\alpha_2 & \beta_1\overline{\alpha_1} \end{pmatrix}, \begin{pmatrix} \alpha'_1 & 0 \\ \beta'_1\alpha'_2 & \beta'_1\overline{\alpha'_1} \end{pmatrix} \right] = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \pmod{\mathfrak{P}},$$

which means: the element  $[G_1, G_2] \equiv 1 \pmod{\mathfrak{P}}$  in  $\mathbb{SL}_1(\mathbb{H})$ . In case  $\mathfrak{i} = \varpi$  or  $\xi\varpi, \mathfrak{j} = \xi$ , an element  $G = \begin{pmatrix} 1 & 0 \\ 0 & \beta_1 \end{pmatrix} \in \mathfrak{U}$  has the form  $\begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \pmod{\mathfrak{P}}$ ; then the derived subgroup of  $U(V, \langle, \rangle)$  degenerates to that of  $\mathbb{SL}_1(\mathbb{H})$ . □

**Remark 6.4.** By Proposition 4.2, arguing for the unitary group of the anisotropic skew hermitian space eventually reduces to above cases.

## 7. THE ANISOTROPIC UNITARY GROUPS III: PRELIMINARIES.

In the following two sections we shall review some Satake's results in [19]. To achieve this aim, in this section we will limit ourself to study the skew hermitian spaces over a matrix ring, and the relative classical groups. Nevertheless it seems more naturally to use Morita equivalent<sup>11</sup> to establish the whole theory, and the reader can refer to [8], [20], and [23].

**7.1. The skew hermitian space over  $M_2(F)$  (cf. [20]).** Let  $M_2(F)$  be the matrix ring over  $F$ . As usual the canonical involution  $-$  on  $M_2(F)$  is given by  $- : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ . Let  $V$  be a free right  $M_2(F)$ -module of rank  $n$  with a basis  $\{x_1, \dots, x_n\}$ , endowed with a skew hermitian form  $\langle, \rangle$  so that  $\langle vA, v'A' \rangle = \overline{A}\langle v, v' \rangle A'$ ,  $\overline{\langle v, v' \rangle} = -\langle v', v \rangle$  for  $v, v' \in V, A, A' \in M_2(F)$ . Consider now the two

<sup>11</sup>In generality, it perhaps involves to define the Clifford algebra referred to a skew field, and to degenerate to its commutative subfield.

canonical orthogonal idempotent elements  $e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$  of  $M_2(F)$  satisfying  $\overline{e_{11}} = e_{22}$ . For  $V$  we let  $V_1 = Ve_{11}$ ,  $V_2 = Ve_{22}$ . Note that  $V_1$  is a linear space over  $F$  of dimension  $2n$  spanned by  $\{x_1e_{11}, x_1e_{21}; \dots; x_ne_{11}, x_ne_{21}\}$  for  $e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ . Define a bilinear form  $(,)$  on the  $F$ -space  $V_1$  by  $(v_1, v'_1) = e_{22}\langle v_1, v'_1 \rangle e_{11}$ ,  $v_1, v'_1 \in V_1$ . By [20, Page 362],  $(,)$  is symmetric and

$$\begin{aligned} \langle v, v' \rangle &= \langle (ve_{11} + v\omega e_{11}\omega), (v'e_{11} + v'\omega e_{11}\omega) \rangle = \langle ve_{11}, v'e_{11} \rangle + \langle ve_{11}, v'\omega e_{11}\omega \rangle \\ &\quad + \overline{\omega}\langle v\omega e_{11}, v'e_{11} \rangle + \overline{\omega}\langle v\omega e_{11}, v'\omega e_{11}\omega \rangle = \begin{pmatrix} -(v\omega e_{11}, v'e_{11}) & -(v\omega e_{11}, v'\omega e_{11}\omega) \\ (ve_{11}, v'e_{11}) & (ve_{11}, v'\omega e_{11}\omega) \end{pmatrix} \end{aligned} \quad (7.1)$$

for  $\omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(F)$ . So the skew hermitian form  $\langle, \rangle$  on  $M_2(F)$  is uniquely determined by the symmetric form  $(,)$  on  $V_1$ , and vice-versa. Let  $U(V, \langle, \rangle)$  (resp.  $SU(V, \langle, \rangle)$ ) be the corresponding unitary (resp. special unitary) group of  $(V, \langle, \rangle)$ . Note that

$$SU(V, \langle, \rangle) = \{f \in \text{End}_{M_2(F)}(V) \mid \langle f(v), f(v') \rangle = \langle v, v' \rangle \text{ for } v, v' \in V \text{ and } \det_F(f) = 1\}.$$

Let  $O(V_1, (,))$  be the corresponding orthogonal group and  $SO(V_1, (,)) = SL(V_1, (,)) \cap O(V_1, (,))$ . Now we have

**Proposition 7.1** ([19, Section 1]). *The mapping  $\vartheta : U(V, \langle, \rangle) \longrightarrow O(V_1, (,)); f \mapsto \vartheta(f)$ , defined by  $\vartheta(f)(v) = f(v)$  for  $v \in V_1$ , is an isomorphism, which sends  $SU(V, \langle, \rangle)$  onto  $SO(V_1, (,))$ .*

*Proof.* As  $\langle f(v), f(v') \rangle = \begin{pmatrix} * & * \\ (f(ve_{11}), f(v'e_{11})) & * \end{pmatrix}$ , the mapping  $\vartheta$  is well-defined. Following [20, Page 362], we define the inverse mapping  $\varsigma$  of  $\vartheta$  by

$$\varsigma(g)(v) = g(ve_{11}) + g(v\omega e_{11}\omega) \quad (7.2)$$

for  $g \in O(V_1, (,))$ ,  $v \in V$ , where the right-hand  $\omega$  sends  $x_ie_{11}$  to  $x_ie_{12}$  and  $x_ie_{21}$  to  $x_ie_{22}$  for each base  $x_i$  of  $V$ . Now we should show that  $\varsigma(g)$  is a  $M_2(F)$ -linear isometry of  $V$ . Taking  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(F)$  and a base  $x_i$  of  $V$ , we then have

$$\begin{aligned} \varsigma(g)(x_iA) &= g(x_iAe_{11}) + g(x_iA\omega e_{11}\omega) = ag(x_ie_{11}) + cg(x_ie_{21}) + bg(x_ie_{11})\omega + dg(x_ie_{21})\omega \\ &= (g(x_ie_{11}) + g(x_i\omega e_{11}\omega)) \cdot (ae_{11} + be_{12} + ce_{21} + de_{22}) = \varsigma(g)(x_i) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{aligned}$$

as required. For  $v, v' \in V$ ,  $g \in O(V_1, (,))$ , utilizing the equations (7.1) and (7.2), we get

$$\begin{aligned} \langle \varsigma(g)(v), \varsigma(g)(v') \rangle &= \begin{pmatrix} -(g(v\omega e_{11}), g(v'\omega e_{11})) & -(g(v\omega e_{11}), g(v'\omega e_{11}\omega)) \\ (g(ve_{11}), g(v'e_{11})) & (g(ve_{11}), g(v'\omega e_{11}\omega)) \end{pmatrix} \\ &= \begin{pmatrix} -(v\omega e_{11}, v'\omega e_{11}) & -(v\omega e_{11}, v'\omega e_{11}\omega) \\ (ve_{11}, v'e_{11}) & (ve_{11}, v'\omega e_{11}\omega) \end{pmatrix} = \langle v, v' \rangle. \end{aligned}$$

On the other hand, under the given basis  $\{x_1, \dots, x_n\}$ , we write  $f \in U(V, \langle, \rangle)$  in terms of a correspond-

ing matrix  $F = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}$ , for  $A_{ij} = \begin{pmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{pmatrix} \in M_2(F)$ . By definition,

$$\theta(f)(x_ie_{11}) = \sum_{k=1}^n x_k A_{ki} e_{11} = \sum_{k=1}^n (a_{ki} x_k e_{11} + c_{ki} x_k e_{21})$$

and

$$\theta(f)(x_i e_{21}) = \sum_{k=1}^n x_k A_{ki} e_{21} = \sum_{k=1}^n (b_{ki} x_k e_{11} + d_{ki} x_k e_{21}).$$

Therefore under the basis  $\{x_1 e_{11}, x_1 e_{21}; \dots, x_n e_{11}, x_n e_{21}\}$  of  $V_1$ ,  $\theta(f)$  corresponds to nothing other than  $F$ , but viewed as an element in  $\text{GL}_{2n, 2n}(F)$  now. It is clear that the second statement holds.  $\square$

**Example 7.2.**  $V = M_2(F)$ ,  $\langle A, A' \rangle := \bar{A} \begin{pmatrix} x & y \\ z & -x \end{pmatrix} A'$  for  $A, A' \in M_2(F)$ . Then  $U(V, \langle \cdot, \cdot \rangle) = \{A \in \text{GL}_2(F) \mid \bar{A} \begin{pmatrix} x & y \\ z & -x \end{pmatrix} A = \begin{pmatrix} x & y \\ z & -x \end{pmatrix}\}$ . In this case,  $V_1 \simeq \left\{ \begin{pmatrix} a \\ c \end{pmatrix} \mid a, c \in F \right\}$ , the symmetric form is defined as  $\left\langle \begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} a' \\ c' \end{pmatrix} \right\rangle = (a, c) \begin{pmatrix} z & -x \\ -x & -y \end{pmatrix} \begin{pmatrix} a' \\ c' \end{pmatrix}$ . The orthogonal group  $O(V_1, \langle \cdot, \cdot \rangle) = \{A \in \text{GL}_2(F) \mid A^t \begin{pmatrix} z & -x \\ -x & -y \end{pmatrix} A = \begin{pmatrix} z & -x \\ -x & -y \end{pmatrix}\}$ . Note that  $\begin{pmatrix} z & -x \\ -x & -y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ z & -x \end{pmatrix}$ , and  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} A^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \bar{A}$  for  $A \in \text{GL}_2(F)$ , so the equation  $A^t \begin{pmatrix} z & -x \\ -x & -y \end{pmatrix} A = \begin{pmatrix} z & -x \\ -x & -y \end{pmatrix}$  is equivalent to  $\left[ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} A^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] \begin{pmatrix} x & y \\ z & -x \end{pmatrix} A = \begin{pmatrix} x & y \\ z & -x \end{pmatrix}$ ; that is the result.

**7.2. Relative classical groups.** Let  $M$  be a vector space over  $F$  of dimension 4. We let  $T(M) = \bigoplus_{0 \leq n \leq 4} M^{\otimes n}$  be the tensor algebra of  $M$ , where  $M^0 = F$  and  $M^{\otimes n}$  is the  $n$ -fold tensor product of  $M$  with itself. Now let  $\mathfrak{I}$  be the two-sided ideal of  $T(M)$  generated by all the elements of the form  $x \otimes x$  for  $x \in M$ . The exterior algebra of  $M$  is understood to be the quotient algebra  $T(M)/\mathfrak{I}$  satisfying the proper universal property, denoted by  $\bigwedge M$ . Clearly  $\bigwedge M$  is a direct sum of its subspaces of degree  $0, 1, \dots, 4$ , i.e.  $\bigwedge M = F \oplus \bigwedge^1 M \oplus \bigwedge^2 M \oplus \bigwedge^3 M \oplus \bigwedge^4 M$ . Obverse that  $\dim_F(\bigwedge^2 M) = 6$  with a basis  $\{x_i \wedge x_j \mid i \neq j\}$ , and  $\dim_F(\bigwedge^4 M) = 1$  with a basis  $\{x_1 \wedge \dots \wedge x_4\}$ . Now the exterior product of two vectors  $x$  and  $y$  of  $\bigwedge^2 M$ , called a *bivector*, can be written in the form  $x \wedge y = Q(x, y)x_1 \wedge \dots \wedge x_4$ . The  $Q(-, -)$  is a non-degenerate symmetric bilinear form on  $\bigwedge^2 M$ . Every  $F$ -automorphism of  $M$ , say  $g \in \text{GL}(M)$ , will give an automorphism  $\tilde{g}$  of  $\bigwedge^2 M$  defined by  $\tilde{g} \cdot x := \sum (g \cdot a) \wedge (g \cdot b)$  for  $x = \sum a \wedge b \in \bigwedge^2 M$ . It is immediate (definition) that  $Q(\tilde{g} \cdot x, \tilde{g} \cdot y) = \det(g)Q(x, y)$ , i.e.  $\tilde{g} \in \text{GO}(\bigwedge^2 M, Q)$ . The reciprocity statement also holds up to multiplicity of scalars. The result is collected in the following theorem:

**Theorem 7.3** ([3, Chapitre IV, §8]). *There is an exact sequence*

$$\begin{array}{ccccccc} 1 & \longrightarrow & F^\times & \longrightarrow & (F^\times \times \text{GL}(M)) & \xrightarrow{\kappa} & \text{GO}^+(\bigwedge^2 M, Q) \longrightarrow 1 \\ & & t & \longmapsto & (t^2, t^{-1}) & & \end{array}$$

where  $\text{GO}^+(\bigwedge^2 M, Q) = \{g \in \text{GO}(\bigwedge^2 M, Q) \mid \det(g) = \lambda(g)^3\}$ , and  $\lambda : \text{GO}(\bigwedge^2 M, Q) \longrightarrow F^\times$  is the similitude character.

**Remark 7.4.** If we choose  $x_{12} = x_1 \wedge x_2$ ,  $x_{34} = x_3 \wedge x_4$ ,  $x_{13} = x_1 \wedge x_3$ ,  $x_{24} = x_2 \wedge x_4$ ,  $x_{14} = x_1 \wedge x_4$ ,

$x_{23} = x_2 \wedge x_3$  to be the basis of  $\bigwedge^2 M$ , then  $(Q(x_{ij}, x_{kl})) = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 0 & -1 & \\ & & -1 & 0 & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix}$ . Through the

mapping  $\kappa$ , the action of  $g = (a_{ij}) \in \text{GL}_4(F) \simeq \text{GL}(M)$  on  $x_k \wedge x_l$  is given by  $g \cdot (x_k \wedge x_l) =$

$\sum_{1 \leq i, j \leq 4} a_{ik} a_{jl} x_i \wedge x_j$ . Following [19], we denote  $g^{(2)}$  to be the matrix in  $\text{GL}_3(M_2(F))$  such that  $g \cdot (x_{12}, x_{34}; x_{13}, x_{24}; x_{14}, x_{23}) = (x_{12}, x_{34}; x_{13}, x_{24}; x_{14}, x_{23})g^{(2)}$ .

### 8. THE ANISOTROPIC UNITARY GROUPS III: SATAKE'S RESULTS.

The purpose of this section is to examine carefully the way that Satake described concretely the group  $\text{SU}_V(F)$  (see below). For the dissimilar way using the language of linear algebraic group schemes, one can consult Gan and Tantonos' paper [7].

Let  $(V = \mathbb{H}(\mathbf{i}) \oplus \mathbb{H}(\mathbf{j}) \oplus \mathbb{H}(\mathbf{l}), \langle, \rangle)$  be a right anisotropic skew hermitian space over  $\mathbb{H}$  of dimension 3 subject to the conditions that (1)  $\{1, \mathbf{i}, \mathbf{j}, \mathbf{l} = \mathbf{i}\mathbf{j} = -\mathbf{j}\mathbf{i}\}$  is a standard basis of  $\mathbb{H}$ ;<sup>12</sup> (2)  $\mathbf{i}^2 = -\alpha, \mathbf{j}^2 = -\beta, \mathbf{l}^2 = -\mathbf{l}^2 = \alpha\beta$ ;<sup>13</sup> (3)  $\mathbf{l} = \mathbf{i}b_0 + \mathbf{j}c_0 + \mathbf{l}d_0$  for  $b_0, c_0, d_0 \in F$ , so  $b_0^2\alpha + c_0^2\beta + d_0^2\alpha\beta = -\alpha\beta$ . Set  $F_1 = F(\mathbf{i})$ ,  $K = F_1(\sqrt{-\beta})$ , and  $\text{Gal}(F_1/F) = \langle \sigma \rangle$ ,  $\text{Gal}(K/F) = \langle \sigma, \tau \rangle$ .

For the right skew hermitian space  $V$  over  $\mathbb{H}$ , one considers its Endomorphism group  $\text{End}_{\mathbb{H}}(V)$ , endowed with a unique adjoint involution  $*$ , such that  $*|_{\mathbb{H}}$  is the canonical involution of  $\mathbb{H}$  and  $\langle f^*(v), v' \rangle = \langle v, f(v') \rangle$  for  $v, v' \in V, f \in \text{End}_{\mathbb{H}}(V)$ . (See [8, Page 42] for the details.)

Now we let  $\mathbf{U}_V$  (resp.  $\text{SU}_V$ ) be the unitary (resp. the special unitary) group of scheme of isometries of  $(\text{End}_{\mathbb{H}}(V), *)$  defined as

$$\mathbf{U}_V(R) = \{a \in (\text{End}_{\mathbb{H}}(V) \otimes_F R)^\times \mid aa^* = 1\}$$

resp.

$$\text{SU}_V(R) = \{a \in (\text{End}_{\mathbb{H}}(V) \otimes_F R)^\times \mid aa^* = 1, \det_R(a) = 1\}$$

for any unital commutative associative algebra  $R$  over  $F$ . Let us denote the isometry group of  $(V, \langle, \rangle)$  by  $\mathbf{U}(V, \langle, \rangle)$ .

**Lemma 8.1.**  $\mathbf{U}(V, \langle, \rangle)$  is isomorphic to  $\text{SU}_V(F)$ .

*Proof.* See [11, Page 21] or [7, Section 2.2]. □

**Lemma 8.2.** *The mapping*

$$s : \mathbb{H} \otimes_F F_1 = \{(1 \otimes 1)a + (\mathbf{i} \otimes 1)b + (\mathbf{j} \otimes 1)c + (\mathbf{l} \otimes 1)d \mid a, b, c, d \in F_1\} \longrightarrow \text{M}_2(F_1),$$

sending  $1 \otimes 1$  to  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $\mathbf{i} \otimes 1$  to  $\begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix}$ ,  $\mathbf{j} \otimes 1$  to  $\begin{pmatrix} 0 & -\beta \\ 1 & 0 \end{pmatrix}$ ,  $\mathbf{l} \otimes 1$  to  $\begin{pmatrix} 0 & -\beta\mathbf{i} \\ -\mathbf{i} & 0 \end{pmatrix}$ , i.e.  $(1 \otimes 1)a + (\mathbf{i} \otimes 1)b + (\mathbf{j} \otimes 1)c + (\mathbf{l} \otimes 1)d$  to  $\begin{pmatrix} a + \mathbf{i}b & -\beta(c + \mathbf{i}d) \\ c - \mathbf{i}d & a - \mathbf{i}b \end{pmatrix}$ , is an isomorphism of algebras. Moreover  $s$  commutes with the corresponding involutions on both sides, and  $s((1 \otimes 1)a^\sigma + (\mathbf{i} \otimes 1)b^\sigma + (\mathbf{j} \otimes 1)c^\sigma + (\mathbf{l} \otimes 1)d^\sigma) = \begin{pmatrix} 0 & -\beta \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a + \mathbf{i}b & -\beta(c + \mathbf{i}d) \\ c - \mathbf{i}d & a - \mathbf{i}b \end{pmatrix}^\sigma \begin{pmatrix} 0 & -\beta \\ 1 & 0 \end{pmatrix}^{-1}$ .

*Proof.* The first statement comes from [20, Page 78, Corollary] and the other ones are immediate. □

Turning to  $V_{F_1} = V \otimes_F F_1$  we obtain a right skew hermitian space over  $\text{M}_2(F_1)$  with the skew hermitian form  $\langle, \rangle_{V_{F_1}}$  given by

$$\langle (A_1, A_2, A_3), (A'_1, A'_2, A'_3) \rangle_{V_{F_1}} = \sum_{v=1}^3 s(\langle s^{-1}(A_v), s^{-1}(A'_v) \rangle)$$

<sup>12</sup>By Corollary 4.5, we assume  $\mathbf{j}^2$  is a prime of  $F$ .

<sup>13</sup>We use the different notion against Satake's:  $\mathbf{i}^2 = -\alpha, \mathbf{j}^2 = -\beta$  instead of  $\mathbf{i}^2 = \alpha, \mathbf{i}^2 = \beta$

$$= \overline{A_1} \begin{pmatrix} \mathfrak{i} & 0 \\ 0 & -\mathfrak{i} \end{pmatrix} A'_1 + \overline{A_2} \begin{pmatrix} 0 & -\beta \\ 1 & 0 \end{pmatrix} A'_2 + \overline{A_3} \begin{pmatrix} b_0 \mathfrak{i} & -\beta(c_0 + d_0 \mathfrak{i}) \\ c_0 - d_0 \mathfrak{i} & -b_0 \mathfrak{i} \end{pmatrix} A'_3$$

for  $A_i, A'_i \in M_2(F_1)$ . By Proposition 7.1 and Example 7.2, the unitary group  $U(V_{F_1}, \langle, \rangle_{V_{F_1}})$  is isomorphic to an orthogonal group  $O(W, (\cdot)_W)$  for

$$W = \left\{ \left( \begin{pmatrix} a_1 \\ c_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ c_2 \end{pmatrix}, \begin{pmatrix} a_3 \\ c_3 \end{pmatrix} \right) \mid a_i, c_i \in F_1 \right\}$$

and

$$\begin{aligned} & \left( \begin{pmatrix} a_1 \\ c_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ c_2 \end{pmatrix}, \begin{pmatrix} a_3 \\ c_3 \end{pmatrix} \right), \left( \begin{pmatrix} a'_1 \\ c'_1 \end{pmatrix}, \begin{pmatrix} a'_2 \\ c'_2 \end{pmatrix}, \begin{pmatrix} a'_3 \\ c'_3 \end{pmatrix} \right) \rangle_W \\ &= (a_1, c_1) \begin{pmatrix} 0 & -\mathfrak{i} \\ -\mathfrak{i} & 0 \end{pmatrix} \begin{pmatrix} a'_1 \\ c'_1 \end{pmatrix} + (a_2, c_2) \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} a'_2 \\ c'_2 \end{pmatrix} + (a_3, c_3) \begin{pmatrix} c_0 - d_0 \mathfrak{i} & -b_0 \mathfrak{i} \\ -b_0 \mathfrak{i} & \beta(c_0 + d_0 \mathfrak{i}) \end{pmatrix} \begin{pmatrix} a'_3 \\ c'_3 \end{pmatrix}. \end{aligned}$$

The corresponding matrix is just  $Q = \text{diag}(Q_1, Q_2, Q_3)$  for  $Q_1 = \begin{pmatrix} 0 & -\mathfrak{i} \\ -\mathfrak{i} & 0 \end{pmatrix}$ ,  $Q_2 = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}$ ,  $Q_3 = \begin{pmatrix} c_0 - d_0 \mathfrak{i} & -b_0 \mathfrak{i} \\ -b_0 \mathfrak{i} & \beta(c_0 + d_0 \mathfrak{i}) \end{pmatrix}$ .

By [19, Page 405], we let  $P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2\mathfrak{i} \end{pmatrix}$ ,  $P_2 = \begin{pmatrix} 1 & \sqrt{-\beta} \\ 1 & -\sqrt{-\beta} \end{pmatrix}$ ,  $P_3 = \begin{pmatrix} -c_0 + d_0 \mathfrak{i} & (b_0 + \sqrt{-\beta})\mathfrak{i} \\ 1 & \frac{(-b_0 + \sqrt{-\beta})\mathfrak{i}}{c_0 - d_0 \mathfrak{i}} \end{pmatrix}$ . Then it can be checked that  $\begin{pmatrix} 0 & -\mathfrak{i} \\ -\mathfrak{i} & 0 \end{pmatrix} = -\frac{1}{2}P_1^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} P_1$ ,  $\begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} = -\frac{1}{2}P_2^t \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} P_2$ , and

$$\begin{pmatrix} c_0 - d_0 \mathfrak{i} & -b_0 \mathfrak{i} \\ -b_0 \mathfrak{i} & \beta(c_0 + d_0 \mathfrak{i}) \end{pmatrix} = -\frac{1}{2}P_3^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} P_3.$$

Notice that  $\sqrt{-\beta} \notin F_1$ . Let  $P = \text{diag}(P_1, P_2, P_3)$ , so  $-\frac{1}{2}P^t \text{diag}(\omega, -\omega, \omega)P = Q$  for  $\omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(K)$ .

Now let  $(W_K, (\cdot)_{W_K})$  be the orthogonal space over  $K$  obtained from  $(W, (\cdot)_W)$  by the scalar extension  $K/F_1$ . We let  $\text{SO}(W, (\cdot)_W)$ ,  $\text{SO}(W_K, (\cdot)_{W_K})$  to be the corresponding special orthogonal groups respectively. In view of Section 7.2, we have

**Lemma 8.3.** *There is an exact sequence*

$$\begin{array}{ccccccc} 1 & \longrightarrow & K^\times & \longrightarrow & C^+(W_K, (\cdot)_{W_K}) & \xrightarrow{\kappa} & \text{SO}(W_K, (\cdot)_{W_K}) \longrightarrow 1 \\ & & t & \longmapsto & (t^2, t^{-1}) & & \end{array}$$

where  $C^+(W_K, (\cdot)_{W_K}) = \{(t, g) \mid g \in \text{GL}_4(K), t \in K^\times \text{ such that } t^2 \det(g) = 1\}$ . The mapping  $\kappa$  is defined by  $(t, g) \mapsto tP^{-1}g^{(2)}P$ , where  $g^{(2)}$  is given in Remark 7.4.

*Proof.* An element  $A \in \text{GL}_3(M_2(K))$  belongs to  $\text{SO}(W_K, (\cdot)_{W_K})$  only if  $A^t Q A = Q$ , which is equivalent to  $A^t [-\frac{1}{2}P^t \text{diag}(\omega, -\omega, \omega)P]A = -\frac{1}{2}P^t \text{diag}(\omega, -\omega, \omega)P$ . By Theorem 7.3 and Remark 7.4, we see  $PAP^{-1} = tg^{(2)}$  for some  $g \in \text{GL}_4(K)$  and  $t \in K^\times$  satisfying  $\det(g)t^2 = 1$ .  $\square$

Now let us determine our concerned group  $\text{SO}(W, (\cdot)_W)$ . Since the above matrix  $Q$  is defined over  $F_1$ , the group  $\text{SO}(W, (\cdot)_W)$  can be identified with the  $\text{Gal}(K/F_1)$ -invariant subgroup of  $\text{SO}(W_K, (\cdot)_{W_K})$ . An element  $(t, g) \in C^+(W_K, (\cdot)_{W_K})$ , under the mapping  $\kappa$ , is  $\tau$ -invariant if  $t^\tau(P^{-1}g^{(2)}P)^\tau = tP^{-1}g^{(2)}P$ , i.e.

$$[P(P^{-1})^\tau][tg^{(2)}]^\tau[P^\tau P^{-1}] = [tg^{(2)}] \quad (8.1)$$

By [19, Page 406], the above isomorphism of  $\mathrm{SO}(W_K, (\cdot, \cdot)_{W_K})$  on the left-hand side of (8.1) up to scalar linear maps, arises from an inner automorphism of  $\mathrm{GL}_4(K)$ . Moreover one has

**Lemma 8.4** ([19, Page 407]).  $P(P^{-1})^\tau = \frac{1}{c_0 - d_0 i} J^{(2)}$  for  $J = \mathrm{diag} \left( \begin{pmatrix} 0 & -c_0 + d_0 i \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ c_0 - d_0 i & 0 \end{pmatrix} \right) \in \mathrm{GL}_4(K)$ .

*Proof.* Note that  $P(P^{-1})^\tau = \mathrm{diag} \left( \begin{pmatrix} 1 & 0 \\ 0 & 2i \end{pmatrix}, \begin{pmatrix} 1 & \sqrt{-\beta} \\ 1 & -\sqrt{-\beta} \end{pmatrix}, \begin{pmatrix} -c_0 + d_0 i & (b_0 + \sqrt{-\beta})i \\ 1 & \frac{(-b_0 + \sqrt{-\beta})i}{c_0 - d_0 i} \end{pmatrix} \right)$   
 $\mathrm{diag} \left( \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2i} \end{pmatrix}, \frac{1}{2\sqrt{-\beta}} \begin{pmatrix} \sqrt{-\beta} & \sqrt{-\beta} \\ -1 & 1 \end{pmatrix}, \frac{1}{2\sqrt{-\beta}i} \begin{pmatrix} \frac{(-b_0 - \sqrt{-\beta})i}{c_0 - d_0 i} & -(b_0 - \sqrt{-\beta})i \\ -1 & -c_0 + d_0 i \end{pmatrix} \right)$   
 $= \mathrm{diag} \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -c_0 + d_0 i \\ \frac{1}{-c_0 + d_0 i} & 0 \end{pmatrix} \right).$

By observation, the given matrix  $J$  is reasonable. Going back to Remark 7.4, we know that  $J$  sends  $x_1$  to  $x_2$ ,  $x_2$  to  $(-c_0 + d_0 i)x_1$ ,  $x_3$  to  $(c_0 - d_0 i)x_4$ , and  $x_4$  to  $-x_3$ . So the transformation  $\frac{J^{(2)}}{c_0 - d_0 i}$  preserves  $x_1 \wedge x_2$ ,  $x_3 \wedge x_4$ , and sends  $x_1 \wedge x_3$  to  $x_2 \wedge x_4$ ,  $x_2 \wedge x_4$  to  $x_1 \wedge x_3$ ,  $x_1 \wedge x_4$  to  $(-c_0 + d_0 i)^{-1}x_2 \wedge x_3$ ,  $x_2 \wedge x_3$  to  $(-c_0 + d_0 i)x_1 \wedge x_4$ ; that is the result.  $\square$

Substituting  $P(P^{-1})^\tau = \frac{1}{c_0 - d_0 i} J^{(2)}$  into the equation (8.1) we get  $[tg^{(2)}]^\tau = t(J^{-1}gJ)^{(2)}$ . So  $t^\tau t^{-1} = (J^{-1}gJ(g^{-1})^\tau)^{(2)}$ . By Lemma 8.3, we have  $\begin{cases} t^\tau t^{-1} = a^2 \\ J^{-1}gJ(g^{-1})^\tau = a \end{cases}$  for some  $a \in K^\times$ . It can be checked that  $N_{K/F_1}(a) = 1$ . By Hilbert's Theorem 90, actually  $a = \frac{b^\tau}{b}$  for some  $b \in K^\times$ . Replacing  $g$  by  $gb^{-1}$ , and  $t$  by  $tb^2$  we can assume  $t^\tau = t$ ,  $J^{-1}gJ = g^\tau$ . Finally we conclude:

**Lemma 8.5.** *There is an exact sequence*

$$1 \longrightarrow F_1^\times \longrightarrow C^+(W, (\cdot, \cdot)_W) \xrightarrow{\kappa} \mathrm{SO}(W, (\cdot, \cdot)_W) \longrightarrow 1,$$

where  $C^+(W, (\cdot, \cdot)_W) = \{(t, g) \mid g \in \mathrm{GL}_4(K), t \in F_1^\times \text{ such that } t^2 \det(g) = 1 \text{ and } J^{-1}gJ = g^\tau\}$ . The mapping  $\kappa$  is defined by  $(t, g) \mapsto tP^{-1}g^{(2)}P$ , where  $g^{(2)}$  is given in Remark 7.4.

Now let us return to the beginning of this subsection.  $\mathrm{U}(V, \langle, \rangle) \simeq \mathrm{SU}_V(F)$  and  $\mathrm{SU}_V(F_1)$  is isomorphic to  $\mathrm{SO}(W, (\cdot, \cdot)_W)$  by the result (Proposition 7.1) of Section 7.1. So in some sense,  $\mathrm{U}(V, \langle, \rangle)$  is isomorphic to the  $\mathrm{Gal}(F_1/F)$ -invariant part of  $\mathrm{SO}(W, (\cdot, \cdot)_W)$ , where the action of  $\sigma$  is given in Lemma 8.2. Now let us describe this explicitly.

By Lemma 8.2, the group  $\mathrm{SU}_V(F)$  is isomorphic to the subgroup of  $\mathrm{SU}(V_{F_1}, \langle, \rangle_{V_{F_1}})$  consisting of those elements  $f$  such that

$$f(\mathrm{diag}(\Delta, \Delta, \Delta) \mathrm{diag}(A_1^\sigma, A_2^\sigma, A_3^\sigma) \mathrm{diag}(\Delta^{-1}, \Delta^{-1}, \Delta^{-1})) = \mathrm{diag}(\Delta_1, \Delta_2, \Delta_3) f(A_1, A_2, A_3)^\sigma \mathrm{diag}(\Delta_1^{-1}, \Delta_2^{-1}, \Delta_3^{-1})$$

for  $A_i \in M_2(F_1)$ ,  $\Delta = \begin{pmatrix} 0 & -\beta \\ 1 & 0 \end{pmatrix}$ . By Proposition 7.1, those  $f$  correspond to the elements  $G$  of  $\mathrm{SO}(W, (\cdot, \cdot)_W)$  such that  $\mathrm{diag}(\Delta, \Delta, \Delta)^{-1} G \mathrm{diag}(\Delta, \Delta, \Delta) = G^\sigma$ . Recall that every element  $G \in \mathrm{SO}(W, (\cdot, \cdot)_W)$  has the form  $tP^{-1}g^{(2)}P$  for some  $(t, g) \in C^+(W, (\cdot, \cdot)_W)$ . Hence the condition is transferred as

$$\mathrm{diag}(\Delta, \Delta, \Delta)^{-1} [tP^{-1}g^{(2)}P] \mathrm{diag}(\Delta, \Delta, \Delta) = [tP^{-1}g^{(2)}P]^\sigma$$

equivalently

$$[P^\sigma \mathrm{diag}(\Delta, \Delta, \Delta)^{-1} P^{-1}] (tg^{(2)}) [P \mathrm{diag}(\Delta, \Delta, \Delta) (P^{-1})^\sigma] = (tg^{(2)})^\sigma.$$

Similarly as before, one has

**Lemma 8.6** ([19, Page 406]).  $P \operatorname{diag}(\Delta, \Delta, \Delta)(P^{-1})^\sigma = \frac{2\sqrt{-\beta}}{b_0 + \sqrt{-\beta}} I^{(2)}$  for  $I = \begin{pmatrix} & -\frac{b_0 + \sqrt{-\beta}}{2} & \\ 1 & & \frac{\sqrt{-\beta}}{2i} \\ & \frac{i(b_0 + \sqrt{-\beta})}{\sqrt{-\beta}} & \end{pmatrix} \in \operatorname{GL}_4(K)$ .

*Proof.*  $P \operatorname{diag}(\Delta, \Delta, \Delta)(P^{-1})^\sigma$

$$\begin{aligned} &= \operatorname{diag} \left( \begin{pmatrix} 1 & 0 \\ 0 & 2i \end{pmatrix}, \begin{pmatrix} 1 & \sqrt{-\beta} \\ 1 & -\sqrt{-\beta} \end{pmatrix}, \begin{pmatrix} -c_0 + d_0 i & (b_0 + \sqrt{-\beta})i \\ 1 & \frac{(-b_0 + \sqrt{-\beta})i}{c_0 - d_0 i} \end{pmatrix} \right) \cdot \operatorname{diag} \left( \begin{pmatrix} 0 & -\beta \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\beta \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\beta \\ 1 & 0 \end{pmatrix} \right) \\ &\quad \cdot \operatorname{diag} \left( \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2i} \end{pmatrix}, -\frac{1}{2\sqrt{-\beta}} \begin{pmatrix} -\sqrt{-\beta} & -\sqrt{-\beta} \\ -1 & 1 \end{pmatrix}, \frac{1}{2\sqrt{-\beta}i} \begin{pmatrix} (b_0 - \sqrt{-\beta})i \\ -1 \end{pmatrix} \begin{pmatrix} (b_0 + \sqrt{-\beta})i \\ -c_0 - d_0 i \end{pmatrix} \right) \\ &= \operatorname{diag} \left( \begin{pmatrix} 0 & \frac{\beta}{2i} \\ 2i & 0 \end{pmatrix}, \begin{pmatrix} \sqrt{-\beta} & 0 \\ 0 & -\sqrt{-\beta} \end{pmatrix}, \frac{1}{2\sqrt{-\beta}i} \begin{pmatrix} 0 & -2\alpha\sqrt{-\beta}(\sqrt{-\beta} + b_0) \\ \frac{2\alpha\sqrt{-\beta}(\sqrt{-\beta} - b_0)}{c_0^2 + \alpha d_0^2} & 0 \end{pmatrix} \right) \\ &= \operatorname{diag} \left( \begin{pmatrix} 0 & \frac{\beta}{2i} \\ 2i & 0 \end{pmatrix}, \begin{pmatrix} \sqrt{-\beta} & 0 \\ 0 & -\sqrt{-\beta} \end{pmatrix}, \begin{pmatrix} 0 & i(\sqrt{-\beta} + b_0) \\ \frac{\beta}{i(\sqrt{-\beta} + b_0)} & 0 \end{pmatrix} \right); \end{aligned}$$

the last matrix sends  $x_1 \wedge x_2$  to  $2ix_3 \wedge x_4$ ,  $x_3 \wedge x_4$  to  $\frac{\beta}{2i}x_1 \wedge x_2$ ,  $x_1 \wedge x_3$  to  $\sqrt{-\beta}x_1 \wedge x_3$ ,  $x_2 \wedge x_4$  to  $-\sqrt{-\beta}x_2 \wedge x_4$ ,  $x_1 \wedge x_4$  to  $\frac{\beta}{i(\sqrt{-\beta} + b_0)}x_2 \wedge x_3$ , and  $x_2 \wedge x_3$  to  $i(\sqrt{-\beta} + b_0)x_1 \wedge x_4$ , which is just  $\frac{2\sqrt{-\beta}}{\sqrt{-\beta} + b_0} I^{(2)}$  since  $I$  maps  $x_1$  to  $x_3$ ,  $x_3$  to  $-\frac{b_0 + \sqrt{-\beta}}{2}x_1$ ,  $x_2$  to  $\frac{i(\sqrt{-\beta} + b_0)}{\sqrt{-\beta}}x_4$ , and  $x_4$  to  $\frac{\sqrt{-\beta}}{2i}x_2$ .  $\square$

Doing likewise as Lemma 8.5 we obtain the following result of Satake:

**Proposition 8.7** ([19, Page 407]). *There exists an exact sequence*

$$1 \longrightarrow F^\times \longrightarrow C^+(V, \langle, \rangle) \xrightarrow{\kappa} \mathbf{SU}_V(F) \longrightarrow 1 \quad (8.2)$$

where  $C^+(V, \langle, \rangle) = \{(t, g) \mid g \in \operatorname{GL}_4(K), t \in F^\times \text{ such that } t^2 \det(g) = 1, J^{-1}gJ = g^\tau, I^{-1}gI = g^\sigma\}$ .

As a consequence we obtain the following result of Satake:

**Theorem 8.8** ([19, Page 407]). (1) *Let  $\mathbb{D}_4 = \{0\} \cup \{g \in \operatorname{GL}_4(K) \mid J^{-1}gJ = g^\tau, I^{-1}gI = g^\sigma\}$ . Then  $\mathbb{D}_4$  is a division algebra over  $F$  of degree 4 (i.e. of dimension 16).*

(2) *Every element  $g \in \mathbb{D}_4$  can be written in the following form:*

$$g = \begin{pmatrix} \alpha_1 & -(c_0 - d_0 i)\alpha_2^\tau & -\frac{b_0 + \sqrt{-\beta}}{2}\alpha_3^\sigma & \frac{\sqrt{-\beta}}{2i}\alpha_4^{\sigma\tau} \\ \alpha_2 & \alpha_1^\tau & \frac{\sqrt{-\beta}}{2i}\alpha_4^\sigma & -\frac{b_0 - \sqrt{-\beta}}{2(c_0 - d_0 i)}\alpha_3^{\sigma\tau} \\ \alpha_3 & -\alpha_4^\tau & \alpha_1^\sigma & -\frac{\sqrt{-\beta}(c_0 + d_0 i)}{i(b_0 + \sqrt{-\beta})}\alpha_2^{\sigma\tau} \\ \alpha_4 & (c_0 - d_0 i)\alpha_3^\tau & \frac{i}{\sqrt{-\beta}}(b_0 + \sqrt{-\beta})\alpha_2^\sigma & \alpha_1^{\sigma\tau} \end{pmatrix}$$

(3)  $\mathbb{D}_4 = \widetilde{K} + A_1 \widetilde{K} + A_2 \widetilde{K} + A_3 \widetilde{K}$  for  $\widetilde{K} = \{\widetilde{\alpha} = \operatorname{diag}(\alpha, \alpha^\tau, \alpha^\sigma, \alpha^{\sigma\tau}) \mid \alpha \in K\}$  and  $A_1 = \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix}$ ,

$$A_2 = \begin{pmatrix} 0 & Y_{12} \\ Y_{21} & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & Z_{12} \\ Z_{21} & 0 \end{pmatrix} \text{ with } X_{11} = \begin{pmatrix} 0 & -(c_0 + d_0 i) \\ 1 & 0 \end{pmatrix},$$



$$X_{22} = \begin{pmatrix} 0 & -\frac{\sqrt{-\beta}(c_0+d_0\mathfrak{i})}{\mathfrak{i}(b_0+\sqrt{-\beta})} \\ \frac{\mathfrak{i}}{\sqrt{-\beta}}(b_0+\sqrt{-\beta}) & 0 \end{pmatrix}, Y_{21} = \begin{pmatrix} 1 & 0 \\ 0 & (c_0-d_0\mathfrak{i}) \end{pmatrix}, Y_{12} = \begin{pmatrix} -\frac{b_0+\sqrt{-\beta}}{2} & 0 \\ 0 & -\frac{b_0-\sqrt{-\beta}}{2(c_0-d_0\mathfrak{i})} \end{pmatrix},$$

$$Z_{12} = \begin{pmatrix} 0 & \frac{\sqrt{-\beta}}{2\mathfrak{i}} \\ \frac{\sqrt{-\beta}}{2\mathfrak{i}} & 0 \end{pmatrix}, Z_{21} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ in } \text{GL}_2(K).^{14}$$

*Proof.* 1) By the following (2),  $\mathbb{D}_4$  is an  $F$ -algebra and every non-zero element of  $\mathbb{D}_4$  is invertible, so the result follows.

2) Let  $I = \begin{pmatrix} 0 & I_{12} \\ I_{21} & 0 \end{pmatrix}$  for  $I_{12} = \begin{pmatrix} -\frac{b_0+\sqrt{-\beta}}{2} & 0 \\ 0 & \frac{\sqrt{-\beta}}{2\mathfrak{i}} \end{pmatrix}$ ,  $I_{21} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\mathfrak{i}(b_0+\sqrt{-\beta})}{\sqrt{-\beta}} \end{pmatrix} \in \text{GL}_2(K)$ , and  $J = \begin{pmatrix} J_{11} & 0 \\ 0 & J_{22} \end{pmatrix}$  for  $J_{11} = \begin{pmatrix} 0 & -c_0+d_0\mathfrak{i} \\ 1 & 0 \end{pmatrix}$ ,  $J_{22} = \begin{pmatrix} 0 & -1 \\ c_0-d_0\mathfrak{i} & 0 \end{pmatrix} \in \text{GL}_2(K)$ . Suppose that  $g = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathbb{D}_4$ . Then  $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} J_{11} & 0 \\ 0 & J_{22} \end{pmatrix} = \begin{pmatrix} J_{11} & 0 \\ 0 & J_{22} \end{pmatrix} \begin{pmatrix} A_{11}^\tau & A_{12}^\tau \\ A_{21}^\tau & A_{22}^\tau \end{pmatrix}$ , and  $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} 0 & I_{12} \\ I_{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_{12} \\ I_{21} & 0 \end{pmatrix} \begin{pmatrix} A_{11}^\sigma & A_{12}^\sigma \\ A_{21}^\sigma & A_{22}^\sigma \end{pmatrix}$   
i.e.  $\begin{cases} A_{11}J_{11} = J_{11}A_{11}^\tau \\ A_{21}J_{11} = J_{22}A_{21}^\tau \\ A_{12}J_{22} = J_{11}A_{12}^\tau \\ A_{22}J_{22} = J_{22}A_{22}^\tau \end{cases}$  and  $\begin{cases} A_{12}I_{21} = I_{21}A_{12}^\sigma \\ A_{11}I_{12} = I_{12}A_{11}^\sigma \\ A_{22}I_{21} = I_{21}A_{22}^\sigma \\ A_{21}I_{12} = I_{21}A_{21}^\sigma \end{cases}$ . By calculation, the above equations tell us that  $g$  has the following forms:

$$g = \begin{pmatrix} x_{11} & -(c_0-d_0\mathfrak{i})x_{21}^\tau & (c_0-d_0\mathfrak{i})x_{24}^\tau & -x_{23}^\tau \\ x_{21} & x_{11}^\tau & x_{23} & x_{24} \\ x_{31} & -x_{41}^\tau & x_{33} & x_{34} \\ x_{41} & (c_0-d_0\mathfrak{i})x_{31}^\tau & -(c_0-d_0\mathfrak{i})x_{34}^\tau & x_{33}^\tau \end{pmatrix}$$

and

$$g = \begin{pmatrix} x_{33}^\sigma & -\frac{b_0+\sqrt{-\beta}}{\sqrt{-\beta}}\mathfrak{i}x_{34}^\sigma & -\frac{b_0+\sqrt{-\beta}}{2}x_{31}^\sigma & -\frac{\sqrt{-\beta}}{2\mathfrak{i}}x_{32}^\sigma \\ -\frac{\sqrt{-\beta}}{\mathfrak{i}(b_0+\sqrt{-\beta})}x_{43}^\sigma & x_{44}^\sigma & \frac{\sqrt{-\beta}}{2\mathfrak{i}}x_{41}^\sigma & \frac{\beta}{2\alpha(b_0+\sqrt{-\beta})}x_{42}^\sigma \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix};$$

this gives the result.

3) This comes from above (2). □

For simplicity, we will identify  $\widetilde{K}$  with  $K$ . Write  $\mathbb{SL}_1(K) = \{k \in K \mid N_{K/F}(k) = 1\}$ .

**Lemma 8.9.** *Under the conditions of the beginning, the quotient group  $F^\times/(F^\times)^2$  is represented by the set  $\{1, -\alpha, \beta, -\alpha\beta\}$ .*

*Proof.* By Corollary 4.5, Lemma 4.4,  $-1 \in N_{F(\mathfrak{i})/F}(F(\mathfrak{i})^\times)$ , or  $-1 \in N_{F(\mathfrak{k})/F}(F(\mathfrak{k})^\times)$ . For example we assume the first case holds. Then  $-\alpha = -1 \cdot \alpha \in N_{F_1/F}(F_1^\times)$ . So  $F^\times = \langle N_{F_1/F}(F_1^\times), -\alpha\beta \rangle = \langle (F^\times)^2, -\alpha, -\alpha\beta \rangle$ . Now  $F^\times = N_{F_1/F}(F_1^\times) \sqcup (-\alpha\beta N_{F_1/F}(F_1^\times)) = (F^\times)^2 \sqcup (-\alpha(F^\times)^2) \sqcup (-\alpha\beta(F^\times)^2) \sqcup (\beta(F^\times)^2)$ . □

**Proposition 8.10.** *Let  $\Xi$  be a coset representatives of  $\mathbb{SL}_1(\mathbb{D}_4)/[\mathbb{SL}_1(\mathbb{D}_4), \mathbb{SL}_1(\mathbb{D}_4)]$ , and let  $\Upsilon = \{(1, 1), (-\alpha, \mathfrak{i}^{-1}), (\beta, \sqrt{-\beta}^{-1}), (-\alpha\beta, \mathfrak{i}^{-1}\sqrt{-\beta}^{-1})\}$ . Then the canonical mapping induced by the sequence (8.2) from  $T = \{\omega\varsigma \mid \omega \in \Xi, \varsigma \in \Upsilon\}$  to  $\text{U}(V, \langle, \rangle)/[\text{U}(V, \langle, \rangle), \text{U}(V, \langle, \rangle)]$  is surjective.*

<sup>14</sup> Note that  $\mathbb{D}_4$  is isomorphic with the algebra  $\widetilde{D}$  in ([19, Page 407]) by sending each  $A_i$  to there  $\omega_i$  and  $\widetilde{K}$  to  $K$  canonically.



*Proof.* Results of Proposition 8.7, show that

$$U(V, \langle, \rangle) / [U(V, \langle, \rangle), U(V, \langle, \rangle)] \simeq C^+(V, \langle, \rangle) / ([C^+(V, \langle, \rangle), C^+(V, \langle, \rangle)] F^\times)$$

by identifying  $F^\times$  with a subgroup of  $C^+(V, \langle, \rangle)$  via the sequence (8.2). Now let

$$N = \{(t, k) \in C^+(V, \langle, \rangle) \mid k \in K^\times, t \in F^\times \text{ such that } N_{K/F}(k)t^2 = 1\}$$

be a subgroup of  $C^+(V, \langle, \rangle)$ . The field extension  $K$  of  $F$  contains  $F(\mathfrak{i})$  and  $F(\sqrt{-\beta})$ , so that

$$N_{K/F}(K^\times) \supseteq \left( N_{F(\mathfrak{i})/F}(F(\mathfrak{i})^\times) \right)^2 \cup \left( N_{F(\sqrt{-\beta})/F}(F(\sqrt{-\beta})^\times) \right)^2 \supseteq (F^\times)^2;$$

thus every element  $g \in C^+(V, \langle, \rangle)$  can be written in the form  $g = (t_1, k_1 g_1)$  for  $k_1 \in K^\times$ , satisfying  $N_{K/F}(k_1)t_1^2 = 1$ , and  $g_1 \in \mathbb{SL}_1(\mathbb{D}_4)$ . This in turn shows that  $\mathbb{SL}_1(\mathbb{D}_4)$  is a normal subgroup of  $C^+(V, \langle, \rangle)$ , and  $C^+(V, \langle, \rangle) = \mathbb{SL}_1(\mathbb{D}_4)N$ . As a consequence, one has

$$[C^+(V, \langle, \rangle), C^+(V, \langle, \rangle)] \simeq [\mathbb{SL}_1(\mathbb{D}_4)N, \mathbb{SL}_1(\mathbb{D}_4)] = [\mathbb{SL}_1(\mathbb{D}_4), \mathbb{SL}_1(\mathbb{D}_4)] \cdot [N, \mathbb{SL}_1(\mathbb{D}_4)],^{15}$$

and

$$C^+(V, \langle, \rangle) / ([C^+(V, \langle, \rangle), C^+(V, \langle, \rangle)] F^\times) \simeq \mathbb{SL}_1(\mathbb{D}_4)N / ([\mathbb{SL}_1(\mathbb{D}_4), \mathbb{SL}_1(\mathbb{D}_4)] F^\times [N, \mathbb{SL}_1(\mathbb{D}_4)]).$$

By observation, the group  $N/\mathbb{SL}_1(K)F^\times$  is represented by the set  $\Upsilon$ ; this ensures the result.  $\square$

**8.1. Anisotropic vector space over  $\mathbb{H}$  of 2 dimension.** Now let  $(V_1 = \mathbb{H}(\mathfrak{j}) \oplus \mathbb{H}(\mathfrak{l}), \langle, \rangle_{V_1})$  be a subspace of  $(V, \langle, \rangle)$ . Therefore the isometry group of  $(V_1, \langle, \rangle_{V_1})$  can be identified with the subgroup

of  $U(V, \langle, \rangle)$  consisting of elements of the form  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in \text{GL}_3(M_2(F))$ , which correspond to those

elements  $(t, g)$  of  $C^+(V, \langle, \rangle)$  of the form  $g = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \in \text{GL}_2(M_2(F))$  with  $\det A_{11} = \det A_{22} = t^{-1}$ .

**Lemma 8.11** ([19, Page 409]).  $D_{F(\mathfrak{i})} = K + A_1 K$  is a quaternion algebra over  $F(\mathfrak{i})$  with the reduced norm  $\text{Nrd}(k_1 + A_1 k_2) = k_1^{1+\tau} + (c_0 - d_0 \mathfrak{i}) k_2^{1+\tau}$  for  $k_1, k_2 \in K$ .

*Proof.* Clearly  $D_{F(\mathfrak{i})}$  is a division algebra over  $F(\mathfrak{i})$  of 2 degree. We decompose  $D_{F(\mathfrak{i})}$  as  $D_{F(\mathfrak{i})} = F(\mathfrak{i}) + \sqrt{-\beta} F(\mathfrak{i}) + A_1 F(\mathfrak{i}) + \sqrt{-\beta} A_1 F(\mathfrak{i})$  with  $(\sqrt{-\beta})^2 = -\beta$ ,  $A_1^2 = -c_0 + d_0 \mathfrak{i}$  and  $\sqrt{-\beta} A_1 = -A_1 \sqrt{-\beta}$ . The reduced norm is just given by  $\text{Nrd}(k_1 + A_1 k_2) = k_1^{1+\tau} + (c_0 - d_0 \mathfrak{i}) k_2^{1+\tau}$  for  $k_1, k_2 \in K$ , which is nonzero unless  $k_1 = k_2 = 0$  by Proposition 8.8 (2). Hence  $D_{F(\mathfrak{i})}$  is isomorphic with the unique quaternion algebra over  $F(\mathfrak{i})$ .  $\square$

The following result of Satake is immediate:

**Proposition 8.12** ([19, Page 409]). *There exists an exact sequence:*

$$1 \longrightarrow F^\times \longrightarrow C^+(V_1, \langle, \rangle_{V_1}) \longrightarrow U(V_1, \langle, \rangle_{V_1}) \longrightarrow 1,$$

where  $C^+(V_1, \langle, \rangle_{V_1}) = \{(t, g) \in C^+(V, \langle, \rangle) \mid t \in F^\times, g \in D_{F(\mathfrak{i})} \text{ and } \text{Nrd}(g)t = 1\}$ .

**Proposition 8.13.** *Let  $\Xi_1$  be a coset representatives of  $\mathbb{SL}_1(\mathbb{D}_{F(\mathfrak{i})}) / [\mathbb{SL}_1(\mathbb{D}_{F(\mathfrak{i})}), \mathbb{SL}_1(\mathbb{D}_{F(\mathfrak{i})})]$ . Then the canonical mapping from  $T_1 = \{\omega \varsigma \mid \omega \in \Omega_1, \varsigma = (1, 1), (-\alpha, \mathfrak{i}^{-1}), (\beta, \sqrt{-\beta}^{-1}), (-\alpha\beta, \mathfrak{i}^{-1} \sqrt{-\beta}^{-1})\}$  to  $U(V_1, \langle, \rangle_1) / [U(V_1, \langle, \rangle_1), U(V_1, \langle, \rangle_1)]$  is surjective.*

*Proof.* The proof is similar as that of Proposition 8.10.  $\square$

<sup>15</sup> $[\mathbb{SL}_1(\mathbb{D}_4)N, \mathbb{SL}_1(\mathbb{D}_4)N] = [N, N] \cdot [\mathbb{SL}_1(\mathbb{D}_4)N, \mathbb{SL}_1(\mathbb{D}_4)N]$ . Given  $n \in N, g, h \in \mathbb{SL}_1(\mathbb{D}_4)$ , the commutator  $[ng, h] = nghg^{-1}n^{-1}h^{-1} = (ngn^{-1}) \cdot (nhn^{-1}) \cdot (ng^{-1}n^{-1}) \cdot (nh^{-1}n^{-1}) \cdot (nhn^{-1}h^{-1}) = [ngn^{-1}, nhn^{-1}] \cdot [n, h]$ .

## 9. THE HOCHSCHILD-SERRE SPECTRAL SEQUENCE

In this section, for the sake of completeness we recall some aspects of spectral sequence of topological group extensions developed by Moore in [12], and [13]. Our purpose is to extend the classical five inflation-restriction exact sequence to six terms in such case, so that one can use it freely in next section.

**9.1. Moore cohomology.** Let  $G$  be a local profinite group, and let  $A$  be a finite abelian group on which  $G$  acts *trivially*. In [12], Moore defined the “normalized” cochain group as  $C^n(G, A) =$  the set of *Borel* functions from  $\underbrace{G \times \cdots \times G}_n$  to  $A$  such that  $f(s_1, \dots, s_n) = 0$ , whenever one of  $s_i$  is equal to 1

for  $n \geq 0$ ;  $C^n(G, A) = 0$  for  $n < 0$ . The coboundary operator  $\delta_n$  is defined as

$$\begin{aligned} \delta_n f(s_1, \dots, s_{n+1}) &= s_1 f(s_2, \dots, s_{n+1}) + \sum_{i=1}^n (-1)^i f(s_1, \dots, s_i s_{i+1}, \dots, s_{n+1}) \\ &\quad + (-1)^{n+1} f(s_1, \dots, s_n), \quad f \in C^n(G, A). \end{aligned}$$

The cohomology groups  $H^*(G, A)$  of the above complex  $C^*(G, A)$  behave well in application. For instance,  $H^2(G, A)$  classifies the central topological extensions of  $G$  by  $A$ . A central topological extension is an exact sequence of topological groups

$$1 \longrightarrow A \xrightarrow{i} \widetilde{G} \xrightarrow{j} G \longrightarrow 1$$

such that  $i(A)$  belongs to the central subgroup of  $\widetilde{G}$  and  $\widetilde{G}/i(A)$  is isomorphic with  $G$  as topological groups.

**9.2. Hochschild-Serre Spectral Sequence.** Now let  $K$  be a normal closed subgroup of  $G$ . To filter  $H^*(G, A)$  with  $G \supseteq K \supseteq 0$ , Moore introduced the following standard filtering  $\{L_j\}$  on the complex  $C^*(G, A)$ :

- (1)  $L_j = \sum_{n=0}^{\infty} L_j \cap C^n(G, A)$ .
- (2)  $L_j \cap C^n(G, A) = 0$ , if  $j > n$ , and  $L_j \cap C^n(G, A) = 0$ , if  $j < 0$ .
- (3) For  $0 \leq j \leq n$ ,  $L_j \cap C^n(G, A)$  is the group of all elements  $f \in C^n(G, A)$  such that  $f(s_1, \dots, s_n)$  depends on  $s_1, \dots, s_{n-j}$ , and the cosets  $s_{n-j+1}K, \dots, s_nK$ .

It is clear that  $\delta(L_j) \subseteq L_j$ . Let  $Z_r^j$  denote the preimage of  $L_{j+r}$  in  $L_j$  and  $E_r^j = Z_r^j / (Z_{r-1}^{j+1} + \delta(Z_{r-1}^{j+1-r}))$ . The group  $E_r^{j,i}$  is just the image of  $Z_r^j \cap C^{i+j}(G, A)$  in  $E_r^j$ . By definition, one has  $E_1^{j,i} \simeq H^{i+j}(L_j/L_{j+1})$ . To use the spectral sequence like [6], we let  $C^j(G/K, C^i(K, A))$  be the group of “normalized”  $j$ -cochains  $f$ ’s on  $G/K$  with values in  $C^i(K, A)$  subject to the condition that  $f(\overline{s_1}, \dots, \overline{s_j})[t_1, \dots, t_i]$  defines a *Borel* function from  $(G/K)^j \times (K)^i$  to  $A$ . On  $C^j(G/K, C^*(K, A))$ , one introduces the natural coboundary operators  $\delta_K^*$ , and obtains the  $i$ -th cohomology group  $H^i(C^j(G/K, C^*(K, A)))$ .

**Lemma 9.1** ([12, Lemma 1.1]).  $E_1^{j,i}$  is isomorphic with  $H^i(C^j(G/K, C^*(K, A)))$  canonically given by  $[f] \mapsto [(\overline{s_1}, \dots, \overline{s_j}) \mapsto f(t_1, \dots, t_i, \overline{s_1}, \dots, \overline{s_j})]$ .

A Borel homomorphism of local compact groups is also continuous, so we have  $H^1(K, A) \simeq \text{Hom}(K, A)$ , and in this situation  $H^1(K, A)$  becomes a topological group equipped with a canonical Borel structure. Along with [6] but taking the Borel structure into account, Moore proved the following result:

**Theorem 9.2.** *There are the isomorphisms:*

- (1)  $E_1^{j,0} \simeq C^j(G/K, A)$ ,  $E_1^{0,i} \simeq H^i(K, A)$ .

(2)  $E_1^{j,1} \simeq C^j(G/K, H^1(K, A))$ , and  $E_2^{j,1} \simeq H^j(G/K, H^1(K, A))$ .

*Proof.* See [12, Page 49 and Page 52, Theorem 1.1].  $\square$

According to [12, Pages 52-53], the composed map  $H^j(G/K, A) \simeq E_2^{j,0} \longrightarrow E_\infty^{j,0} \hookrightarrow H^j(G, A)$  is *inflation* from  $G/K$  to  $G$ ,  $j = 1, 2$  and the composed map  $H^i(G, A) \twoheadrightarrow E_\infty^{0,i} \hookrightarrow E_2^{0,i} \simeq H^i(K, A)$  is *restriction* from  $G$  to  $H$ . By [6, Page 130, Remark] we have the following inflation-restriction sequence

$$0 \longrightarrow H^1(G/K, A) \xrightarrow{\text{inf}_1} H^1(G, A) \xrightarrow{\text{res}_1} H^1(K, A)^G \xrightarrow{d_2} H^2(G/K, A) \xrightarrow{\text{inf}_2} H^2(G, A). \quad (9.1)$$

**9.3. Six terms.** But for our purpose, we shall extend above long exact sequence at least to six terms. Now let  $H^2(G, A)_1$  denote the kernel of restriction from  $H^2(G, A)$  to  $H^2(K, A)$ , which is isomorphic with  $H^2(L_1)$ . Recall that the coimage of above  $d_2$  is  $E_\infty^{2,0}$ , which is isomorphic with  $H^2(L_2)$ . Note that  $H^2(L_1)/H^2(L_2)$  is isomorphic with  $E_\infty^{1,1}$ . Now  $\delta : E_2^{1,1} \longrightarrow E_2^{3,-1}$  is null, so there is an embedding  $E_\infty^{1,1} \hookrightarrow E_2^{1,1} \simeq H^1(G/K, H^1(K, A))$ . Hence we conclude that there is an exact sequence

$$0 \longrightarrow H^1(G/K, A) \xrightarrow{\text{inf}_1} H^1(G, A) \xrightarrow{\text{res}_1} H^1(K, A)^G \xrightarrow{d_2} H^2(G/K, A) \xrightarrow{\text{inf}_2} H^2(G, A)_1 \xrightarrow{p} H^1(G/K, H^1(K, A)). \quad (9.2)$$

Remark that the above sequence coming from certain spectral sequence is functorial over the pair  $(G, K)$ .

**9.4. Explicit expression.** For convenient use, let us describe explicitly the map  $p$  in terms of cocycles by following [6]. Let  $[c] \in H^2(G, A)_1$  such that the restriction of the cocycle  $c$  to  $K \times K$  is trivial. By definition, we have

$$c(s_2, s_3) - c(s_1 s_2, s_3) + c(s_1, s_2 s_3) - c(s_1, s_2) = 0, \quad s_1, s_2, s_3 \in G \quad (9.3)$$

We choose a set of representatives  $\Omega = \{s^* \in G\}$  for  $G/K$  such that the restriction of the morphism  $G \longrightarrow G/K$  to  $\Omega$  is a Borel isomorphism, and  $1_G \in \Omega$ . For  $s = s^* t$ ,  $t \in K$ , define  $h(s) = c(s^*, t)$ . Note that

$$\delta_1 h(s, t') = h(t') - h(st') + h(s) = -c(s^*, tt') + c(s^*, t), \quad t' \in K \quad (9.4)$$

Consider

$$c^*(s, s_1) = c(s, s_1) + \delta_1 h(s, s_1), \quad s = s^* t, s_1 \in G.$$

Then  $c^*(s, t') = (c + \delta_1 h)(s, t')$  is zero by (9.3) and (9.4). As  $\delta_2 c^* = \delta_2(c + \delta_1 h) = 0$ , we have

$$0 = \delta_2 c^*(s_1, s^*, t) = c^*(s^*, t) - c^*(s_1 s^*, t) + c^*(s_1, s^* t) - c^*(s_1, s^*).$$

Finally, we get

$$c^*(s_1, s) = c^*(s_1, s^* t) = c^*(s_1, s^*),$$

which depends only on  $s_1$  and the coset  $sK$ . In this way, by replacing  $c$  with  $c^*$ , the map  $p$  is given by

$$p([c^*]) : sK \longmapsto (t \longrightarrow c^*(t, s)), \quad t \in K, s \in G.$$

According to [6, Pages 121-122], above  $p$  is well-defined. <sup>16</sup>

<sup>16</sup>In fact,  $c^*(t_1 t_2, s) = c^*(t_1, t_2 s) - c^*(t_1, t_2) + c^*(t_2, s) = c^*(t_1, s) + c^*(t_2, s)$  for  $t_1, t_2 \in K, s \in G$ , and  $c^*(t, s_1 s_2) = c^*(ts_1, s_2) - c^*(s_1, s_2) + c^*(t, s_1) = c^*(s_1^{-1} t s_1, s_2) + c^*(t, s_1) = s_1 \cdot c^*(t, s_2) + c^*(t, s_1)$  for  $t \in K, s_1, s_2 \in G$ , by using the equality:  $0 = c^*(s_1^{-1} s_1, s_2) = c^*(s_1^{-1}, s_1 s_2) - c^*(s_1^{-1}, s_1) + c^*(s_1, s_2)$ .

## 10. A CRITERION

We follow the notations of Section 1.1. Until the end of this section, we will let  $\Lambda_\Gamma := \{\lambda_1(g_1) = \lambda_2(g_2)^{-1} | (g_1, g_2) \in \Gamma\}$ . Then there exists a short exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & U(W_1) \times U(W_2) & \longrightarrow & \Gamma & \xrightarrow{\lambda} & \Lambda_\Gamma \longrightarrow 1 \\ & & & & (g_1, g_2) & \longmapsto & \lambda_1(g_1) \end{array}$$

Applying the given map  $\iota$  in Section 1, we obtain

$$0 \longrightarrow \iota(U(W_1) \times U(W_2)) \longrightarrow \iota(\Gamma) \longrightarrow \iota(\Gamma)/\iota(U(W_1) \times U(W_2)) \longrightarrow 1.$$

By abuse of notation, we write  $\iota(\Lambda_\Gamma)$  for  $\iota(\Gamma)/\iota(U(W_1) \times U(W_2))$ . By Hochschild-Serre spectral sequence (cf. Section 9), there exists the following long exact sequence:

$$\begin{aligned} 0 \longrightarrow \text{Hom}(\iota(\Lambda_\Gamma), \mu_8) &\longrightarrow \text{Hom}(\iota(\Gamma), \mu_8) \longrightarrow \text{Hom}(\iota(U(W_1) \times U(W_2)), \mu_8)^{\iota(\Gamma)} \longrightarrow H^2(\iota(\Lambda_\Gamma), \mu_8) \\ &\longrightarrow H^2(\iota(\Gamma), \mu_8)_1 \longrightarrow H^1(\iota(\Lambda_\Gamma), H^1(\iota(U(W_1) \times U(W_2)), \mu_8)) \end{aligned}$$

Now let  $[c_{Rao}] \in H^2(\text{Sp}(W), \mu_8)$  be the unique nontrivial class of order 2 and  $[c]$  its restriction to  $\iota(\Gamma)$ . By Theorem 1.1, the restriction of  $[c]$  to  $\iota(U(W_1) \times U(W_2))$  is trivial apart from the exceptional case there.

Let  $\Gamma_1$  be a closed subgroup of  $\Gamma$  satisfying the following four conditions:

- (C1)  $\lambda : \Gamma_1 \longrightarrow \Lambda_\Gamma$  is surjective;
- (C2)  $\text{Hom}(\iota(\Gamma_1), \mu_8) \longrightarrow \text{Hom}\left(\iota(U(W_1) \times U(W_2))_0, \mu_8\right)^{\iota(\Gamma_1)}$  is surjective, where  $\iota(U(W_1) \times U(W_2))_0 = \iota(\Gamma_1) \cap \iota(U(W_1) \times U(W_2))$ ; <sup>17</sup>
- (C3) Under the restriction map  $H^2(\iota(\Gamma), \mu_8) \longrightarrow H^2(\iota(\Gamma_1), \mu_8)$  the image of  $[c]$  is trivial;
- (C4) There is a set  $\Omega_\nu$  of representatives for  $\iota(U(W_\nu))/[i(U(W_\nu)), \iota(U(W_\nu))]$  such that  $[\Omega_\nu]^{\iota(\Gamma_1)}$  belongs to  $P(Y_\nu)$  for a Lagrangian  $Y_\nu$  of  $W$ , each  $\nu = 1, 2$ .

**Lemma 10.1.** *If one subgroup  $\Gamma_1$  of  $\Gamma$  satisfies above conditions (C1)—(C4), then the exact sequence*

$$1 \longrightarrow \mu_8 \longrightarrow \bar{\Gamma} \longrightarrow \iota(\Gamma) \longrightarrow 1$$

*is splitting at  $\iota(\Gamma)$ .*

*Proof.*  $\iota(\Gamma)$  is identified with  $\iota(\Gamma_1) \cdot \iota(U(W_1) \times U(W_2))$ , so

$$\iota(\Gamma)/\iota(U(W_1) \times U(W_2)) \simeq \iota(\Gamma_1)/\iota(U(W_1) \times U(W_2))_0 \simeq \iota(\Lambda_\Gamma),$$

and there is a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \iota(U(W_1) \times U(W_2))_0 & \longrightarrow & \iota(\Gamma_1) & \longrightarrow & \iota(\Lambda_\Gamma) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \iota(U(W_1) \times U(W_2)) & \longrightarrow & \iota(\Gamma) & \longrightarrow & \iota(\Lambda_\Gamma) \longrightarrow 1 \end{array}$$

By the six-term exact sequence (9.2) in Section 9.3, we obtain the following commutative diagram of long exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^2(\iota(\Lambda_\Gamma), \mu_8) & \xrightarrow{\alpha_1} & H^2(\iota(\Gamma_1), \mu_8)_1 & \longrightarrow & H^1\left(\iota(\Lambda_\Gamma), H^1(\iota(U(W_1) \times U(W_2))_0, \mu_8)\right) \longrightarrow \cdots \\ & & \uparrow & & \uparrow l & & \uparrow \\ \cdots & \longrightarrow & H^2(\iota(\Lambda_\Gamma), \mu_8) & \xrightarrow{\alpha} & H^2(\iota(\Gamma), \mu_8)_1 & \xrightarrow{p} & H^1\left(\iota(\Lambda_\Gamma), H^1(\iota(U(W_1) \times U(W_2)), \mu_8)\right) \longrightarrow \cdots \end{array}$$

<sup>17</sup> The kernel of  $\iota : U(W_1) \times U(W_2) \longrightarrow \text{Sp}(W_1 \otimes W_2)$  is just  $A_2 = \{(1, 1), (-1, -1)\}$ . When  $A_2$  belongs to  $\Gamma_1$ , there is  $\iota(\Gamma_1) \cap \iota(U(W_1) \times U(W_2)) = \iota(\Gamma_1 \cap U(W_1) \times U(W_2))$ . In the general case, one can add  $A_2$  to  $\Gamma_1$  to avoid the problem.

Taking out the kernel  $\{(1, 1), (-1, -1)\}$ , it is possible to divide the horizontal arrow  $p$  into  $p_1, p_2$ , for  $p_\nu : H^2(\iota(\Gamma), \nu_8) \rightarrow H^1(\iota(\Gamma), H^1(\iota(U(W_\nu)), \mu_8))$ ,  $\nu = 1, 2$ . Now let  $c_\nu$  be a 2-cocycle constructed in [11, Page 55, Théorème], associated to  $Y_\nu$  and  $\psi$ , which is a Borel function from  $\mathrm{Sp}(W) \times \mathrm{Sp}(W)$  to  $\mu_8$ . Under restriction from  $H^2(\iota(\Gamma), \mu_8)$  to  $H^2(\iota(U(W_1) \times U(W_2)), \mu_8)$ , the image of  $[c_\nu]$  is trivial; that means  $[c_\nu]$  lies in  $H^2(\iota(\Gamma), \mu_8)_1$ . So there is a Borel function  $f$  from  $\iota(U(W_1) \times U(W_2))$  to  $\mu_8$  such that

$$c_\nu(t_1, t_2) = f(t_1 t_2) f(t_1)^{-1} f(t_2)^{-1}, \quad t_1, t_2 \in \iota(U(W_1) \times U(W_2)).$$

Note that for  $t_0 \in \Omega_\nu \subseteq P(Y_\nu)$ ,  $t \in \iota(U(W_1) \times U(W_2))$ , we have  $f(t_0 t) = f(t_0) f(t) = f(t t_0)$ . According to [12, Definition 1.2], we choose a set of representatives  $\Delta = \{s^* \in \iota(\Gamma_1)\}$  for  $\iota(\Gamma)/\iota(U(W_1) \times U(W_2))$  subject to the conditions that the restriction of  $\iota(\Gamma) \rightarrow \iota(\Gamma)/\iota(U(W_1) \times U(W_2))$  to  $\Delta$  is a Borel isomorphism, and  $\Delta$  contains the identity element of  $\iota(\Gamma)$  or  $\iota(\Gamma_1)$ . Now let  $f$  extend to a Borel function of  $\iota(\Gamma)$  by taking the trivial value outside  $\iota(U(W_1) \times U(W_2))$ . We replace  $c_\nu$  with  $c'_\nu = c_\nu \cdot \delta f$ , i.e. now  $c'_\nu(s_1, s_2) = c_\nu(s_1, s_2) f(s_1) f(s_2) f(s_1 s_2)^{-1}$  for  $s_1, s_2 \in \iota(\Gamma)$ . It is immediate that  $c'_\nu(t, t') = 1$  for  $t, t' \in \iota(U(W_1) \times U(W_2))$ . Following Section 9.4, we define a Borel function  $h$  of  $\iota(\Gamma)$  as

$$h(s) = c'_\nu(s^*, t), \quad s = s^* t \in \iota(\Gamma).$$

Consider now the cocycle  $c_\nu^* = c'_\nu(\delta h)$ ; then the map  $p_\nu$  is given by

$$p_\nu([c_\nu]) : s^* \mapsto (t \rightarrow c_\nu^*(t, s^*)), \quad t \in \iota(U(W_\nu)), s^* \in \Delta.$$

Notice that  $p_\nu([c_\nu])(s^*)$  belongs to  $\mathrm{Hom}(\iota(U(W_\nu)), \mu_8)$ , so it depends only on the values at those  $t_0 \in \Omega_\nu$ . Now

$$c_\nu^*(t_0, s^*) = c'_\nu(t_0, s^*) [\delta h(t_0, s^*)] = c'_\nu(t_0, s^*) h(t_0 s^*) = c'_\nu(t_0, s^*) c'_\nu(s^*, (s^*)^{-1} t_0 s^*);$$

by definition the last term is equal to  $f(t_0) f^{s^*}(t_0)^{-1}$ , which is a coboundary with respect to  $H^1(\iota(\Gamma), \mathrm{Hom}(\iota(U(W_\nu)), \mu_8))$ . Hence the image of  $[c_\nu]$  under the map  $p_\nu$  is trivial, and  $[c_1] = [c_2] = \alpha([d])$  for some  $[d] \in H^2(\iota(\Lambda_\Gamma), \mu_8)$ . By assumption,  $l([c_\nu]) = 0$ . Then  $\alpha_1([d]) = l([c_\nu]) = 0$ ; as  $\alpha_1$  is injective, we get  $[d] = 0$ , and then  $[c_\nu] = 0$ , so the result follows.  $\square$

In the following Sections 11—17, we shall prove Theorem A. Our main tool is above lemma, and it reduces to find certain suitable subgroup  $\Gamma_1$  of  $\Gamma$  satisfying the desired conditions (C1)—(C4).

## 11. THE PROOF OF THE MAIN THEOREM: THE EXCEPTIONAL CASE.

As an example we prove Theorem A in the exceptional case mentioned in Section 1. Let us gather some lemmas, so cited freely in next Sections.

**Lemma 11.1.** *Let  $|k_F|$  denote the cardinality of the residue field  $k_F$  of  $F$ . Then  $-1 \in (F^\times)^2$  if  $|k_F| \equiv 1 \pmod{4}$  and  $-1 \notin (F^\times)^2$  if  $|k_F| \equiv 3 \pmod{4}$ .*

*Proof.* See [10, Page 154, Corollary 2.6].  $\square$

**Lemma 11.2.** *If  $-1 \notin (F^\times)^2$ , then  $-1$  is a norm in the extension of  $F(\sqrt{-1})/F$ .*

*Proof.* Because  $(-1, -1)_p = 1$  (see [21, Pages 211-212]), where  $(, )_p$  is the Hilbert symbol.  $\square$

Until the end of this section, we will let  $\{1, \mathfrak{i}, \mathfrak{j}, \mathfrak{k}\}$  be a standard basis of  $\mathbb{H}$  with  $\mathfrak{i}^2 = -\alpha, \mathfrak{j}^2 = -\beta$ , such that  $\mathfrak{i}\mathfrak{j} = -\mathfrak{j}\mathfrak{i} = \mathfrak{k}$ . Set  $F_1 = F(\mathfrak{i}), F_2 = F(\mathfrak{j}), F_3 = F(\mathfrak{k})$ . Then:

**Lemma 11.3.** (1) *The group  $F^\times/(F^\times)^2$  has order 4.*  
 (2)  $-1 \in \cup_{a=1}^3 N_{F_a/F}(F_a^\times)$ .

- (3) If  $-1 \in N_{F_1/F}(F_1^\times)$ , then  $F^\times = \langle (F^\times)^2, -\alpha, -\beta \rangle = \langle (F^\times)^2, -\alpha, \beta \rangle$ , and  
 $F^\times / (F^\times)^2 = \{(F^\times)^2, -\alpha(F^\times)^2, -\beta(F^\times)^2, \alpha\beta(F^\times)^2\}$ .

*Proof.* (1) is well-known. (2) comes from the fact that  $(-1, -\alpha)_p \cdot (-1, -\beta)_p \cdot (-1, -\alpha\beta)_p = (-1, -1)_p = 1$ . For (3), see Lemma 8.9.  $\square$

When  $-1 \notin (F^\times)^2$  but belongs to  $N_{F_1/F}(F_1^\times)$ , we can assume above  $i^2 = -1$ . In this case, for simplicity, we take some  $a_{-1}, b_{-1} \in F^\times$  such that  $e_{-1} = a_{-1} + b_{-1}i \in F_1^\times$  satisfies  $\text{Nrd}(e_{-1}) = a_{-1}^2 + b_{-1}^2 = -1$ .

**11.1. The exceptional case I.** We follow the notations introduced in Section 1.1. Suppose now that  $W_1 = \mathbb{H}(i)$ , the skew hermitian form  $\langle, \rangle_1$  is defined as  $\langle d_1, d'_1 \rangle_1 = \overline{d_1} i d'_1$ , for vectors  $d_1, d'_1 \in \mathbb{H}$ , and  $W_2 = \mathbb{H}$ ,  $\langle d_2, d'_2 \rangle_2 = d_2 \overline{d'_2}$ , for vectors  $d_2, d'_2 \in \mathbb{H}$ .

**Lemma 11.4.** *If the space  $W_1$  is endowed with the distinct  $F$ -symplectic form  $\langle, \rangle_{1,F} = \text{Trd}(\langle, \rangle_1)$ , then the canonical mapping*

$$\theta : W = W_1 \otimes_{\mathbb{H}} W_2 \longrightarrow W_1; \quad w_1 \otimes a \longmapsto w_1 \cdot a,$$

*defines an isometry between  $(W, \langle, \rangle)$  and  $(W_1, \langle, \rangle_{1,F})$ .*

*Proof.* Recalling that, for  $w_1, w'_1 \in W_1$ ,  $d, d' \in \mathbb{H}$ , we have  $\langle w_1 d, w'_1 d' \rangle_1 = \overline{d} \langle w_1, w'_1 \rangle_1 d'$ , and  $\langle w_1 \otimes d, w'_1 \otimes d' \rangle = \text{Trd}(\langle w_1, w'_1 \rangle_1 \tau(\langle d, d' \rangle_2)) = \text{Trd}(\langle w_1, w'_1 \rangle_1 d' \overline{d}) = \text{Trd}(\overline{d} \langle w_1, w'_1 \rangle_1 d') = \text{Trd}(\langle w_1 d, w'_1 d' \rangle_1)$ , so the result follows.  $\square$

Let us explain why  $(U(W_1), U(W_2))$  is not an irreducible reductive dual pair (cf. [11, Page 15]).  $W_1 = \mathbb{H}(i)$  becomes a skew hermitian space over  $F_1$  of dimension 2, when decomposed as  $W_1 = F_1 \oplus F_1 j$ , and endowed with the form  $\langle, \rangle_{1,F_1}$ , defined as

$$\langle \alpha_0 + \alpha_1 j, \alpha'_0 + \alpha'_1 j \rangle_{1,F_1} = \overline{\alpha_0} i \alpha'_0 + (-\beta) \alpha_1 i \overline{\alpha'_1}, \quad \alpha_i, \alpha'_i \in F_1.$$

By observation, we have

**Lemma 11.5.**  $\langle, \rangle_{1,F} = \text{Tr}_{F_1/F}(\langle, \rangle_{1,F_1})$

Luckily through above mapping  $\theta$ , the groups

$$U(W_1, \langle, \rangle_1) = \mathbb{SL}_1(F_1), U(W_2, \langle, \rangle_2) = \mathbb{SL}_1(\mathbb{H})$$

now act on  $W_1$ , explained specifically by

$$[\mu, \alpha_0 + \alpha_1 j] = \mu \cdot (\alpha_0 + \alpha_1 j) = \mu \alpha_0 + \mu \alpha_1 j,$$

and

$$[\alpha_0 + \alpha_1 j, \mu_0 + \mu_1 j] = (\alpha_0 + \alpha_1 j) \cdot (\mu_0 + \mu_1 j) = (\mu_0 \alpha_0 - \beta \overline{\mu_1} \alpha_1) + (\mu_1 \alpha_0 + \overline{\mu_0} \alpha_1) j$$

respectively, for  $\mu \in U(W_1, \langle, \rangle_1)$ ,  $\mu_0 + \mu_1 j \in U(W_2, \langle, \rangle_2)$  and  $\alpha_0 + \alpha_1 j \in W_1$ . As a consequence we can claim that the images of  $U(W_1, \langle, \rangle_1)$  and  $U(W_2, \langle, \rangle_2)$  in  $\text{Sp}(W_1, \langle, \rangle_{1,F})$  sit in  $U(W_1, \langle, \rangle_{1,F_1})$  well.

Notice that the element  $(\mu, \nu) \in \mathbb{SL}_1(F_1) \times \mathbb{SL}_1(F_1)$ , when acting on  $W_1$  as

$$(\mu, \nu) \cdot (\alpha_0 + \alpha_1 j) := \mu \alpha_0 + \nu \alpha_1 j,$$

also belongs to  $U(W_1, \langle, \rangle_{1,F_1})$ , and may not to  $U(W_2, \langle, \rangle_2)$ , but commutes with every element of  $U(W_1, \langle, \rangle_1)$ .

Now let  $\overline{U(W_1, \langle, \rangle_{1,F_1})}$  be the inverse image of  $U(W_1, \langle, \rangle_{1,F_1})$  in  $\overline{\text{Sp}(W_1, \langle, \rangle_{1,F})}$ .

**Lemma 11.6.** *The exact sequence  $1 \longrightarrow \mu_8 \longrightarrow \overline{U(W_1, \langle, \rangle_{1,F_1})} \longrightarrow U(W_1, \langle, \rangle_{1,F_1}) \longrightarrow 1$  is splitting.*

*Proof.* This is one special case of Theorem 1.1.  $\square$

**11.2. The exceptional case II.** We now take a symplectic basis  $\mathcal{A} = \{1, \mathfrak{j}; -\frac{\mathfrak{i}}{2\alpha}, \frac{\mathbb{k}}{2\alpha\beta}\}$  of  $(W_1, \langle, \rangle_{1,F})$  so that  $\text{Trd}(1 \cdot \mathfrak{i} \cdot (-\frac{\mathfrak{i}}{2\alpha})) = 1 = \text{Trd}(\bar{\mathfrak{j}} \cdot \mathfrak{i} \cdot \frac{\mathbb{k}}{2\alpha\beta})$ , etc. Under such basis the element  $\mu = a_0 + b_0\mathfrak{i} \in U(W_1, \langle, \rangle_1)$ , satisfying  $a_0^2 + b_0^2\alpha = 1$ , acts on  $(W_1, \langle, \rangle_{1,F})$  in terms of the matrix<sup>18</sup>

$$G_\mu = \begin{pmatrix} a_0 & 0 & \frac{b_0}{2} & 0 \\ 0 & a_0 & 0 & -\frac{b_0}{2\beta} \\ -2\alpha b_0 & 0 & a_0 & 0 \\ 0 & 2\alpha\beta b_0 & 0 & a_0 \end{pmatrix}$$

**Lemma 11.7.** *Let  $A = \begin{pmatrix} 1 & 0 \\ A_{21} & 1 \end{pmatrix} \in \text{GL}_2(\text{M}_2(F))$  with  $A_{21} = \begin{pmatrix} 0 & 4\alpha \\ \beta & 0 \end{pmatrix}$ . Then  $A^{-1}G_\mu A = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \text{GL}_2(\text{M}_2(F))$ .*

*Proof.* Writing  $G_\mu = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ , we have

$$A^{-1}G_\mu A = \begin{pmatrix} 1 & 0 \\ -A_{21} & 1 \end{pmatrix} \cdot \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ A_{21} & 1 \end{pmatrix} = \begin{pmatrix} * & * \\ X_{21} - A_{21}X_{12}A_{21} & * \end{pmatrix}.$$

As  $A_{21}X_{12}A_{21} = \begin{pmatrix} 0 & 4\alpha \\ \beta & 0 \end{pmatrix} \begin{pmatrix} \frac{b_0}{2} & 0 \\ 0 & -\frac{b_0}{2\beta} \end{pmatrix} \begin{pmatrix} 0 & 4\alpha \\ \beta & 0 \end{pmatrix} = \begin{pmatrix} -2\alpha b_0 & 0 \\ 0 & 2\alpha\beta b_0 \end{pmatrix} = X_{21}$ , we obtain the result.  $\square$

As a consequence we obtain

**Proposition 11.8.** *Let  $\mathfrak{e}_1 = 1 + \frac{1}{2\alpha}\mathbb{k}$ ,  $\mathfrak{e}_2 = -2\mathfrak{i} + \mathfrak{j}$  be two vectors in  $W_1$ , and  $Y_1 = \text{Span}\{\mathfrak{e}_1, \mathfrak{e}_2\}$ . Then:*

- (1)  $Y_1$  is a Lagrangian subspace of  $(W_1, \langle, \rangle_{1,F})$ ;
- (2) The image  $\iota(U(W_1, \langle, \rangle_1))$  in  $\text{Sp}(W_1, \langle, \rangle_{1,F})$  belongs to  $P(Y_1)$ .

*Proof.* Moving the basis  $\mathcal{A}$  of  $(W_1, \langle, \rangle_{1,F})$  to another one  $(\mathfrak{e}_1, \mathfrak{e}_2; -\frac{1}{2\alpha}\mathfrak{i}, \frac{1}{2\alpha\beta}\mathbb{k}) = (1, \mathfrak{j}; -\frac{\mathfrak{i}}{2\alpha}, \frac{1}{2\alpha\beta}\mathbb{k})A$ , by standard linear transformation, we see that  $\mu(\mathfrak{e}_1, \mathfrak{e}_2; -\frac{1}{2\alpha}\mathfrak{i}, \frac{1}{2\alpha\beta}\mathbb{k}) = (\mathfrak{e}_1, \mathfrak{e}_2; -\frac{1}{2\alpha}\mathfrak{i}, \frac{1}{2\alpha\beta}\mathbb{k})A^{-1}G_\mu A = (\mathfrak{e}_1, \mathfrak{e}_2; -\frac{1}{2\alpha}\mathfrak{i}, \frac{1}{2\alpha\beta}\mathbb{k}) \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ , which yields the results.  $\square$

To begin considering the group  $U(W_2, \langle, \rangle_2)$ , let us invoke the results of Corollary 4.5 (1): Up to isometry,  $\mathfrak{i}$  should equal to one of  $\xi$ ,  $\varpi$ , and  $\xi\varpi$ .

**11.3. Case I.** Suppose  $\mathfrak{i} = \xi$ , with  $k_{F_1} = k_{\mathbb{H}}$ . By Lemma 5.3,

$$U(W_2, \langle, \rangle_2)/[U(W_2, \langle, \rangle_2), U(W_2, \langle, \rangle_2)] \simeq \mathbb{SL}_1(F_1)/[\mathbb{SL}_1(F_1) \cap (1 + \mathfrak{P})].$$

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<sup>18</sup>  $(a_0 + b_0\mathfrak{i}) \cdot (1, \mathfrak{j}, -\frac{\mathfrak{i}}{2\alpha}, \frac{\mathbb{k}}{2\alpha\beta}) = (a_0 + b_0\mathfrak{i}, a_0\mathfrak{j} + b_0\mathbb{k}, \frac{b_0}{2} - \frac{a_0}{2\alpha}\mathfrak{i}, -\frac{b_0}{2\beta}\mathfrak{j} + \frac{a_0}{2\alpha\beta}\mathbb{k}) = (1, \mathfrak{j}, -\frac{\mathfrak{i}}{2\alpha}, \frac{\mathbb{k}}{2\alpha\beta}) \cdot \begin{pmatrix} a_0 & 0 & \frac{b_0}{2} & 0 \\ 0 & a_0 & 0 & -\frac{b_0}{2\beta} \\ -2\alpha b_0 & 0 & a_0 & 0 \\ 0 & 2\alpha\beta b_0 & 0 & a_0 \end{pmatrix}$



Similarly, under the offered symplectic basis  $\mathcal{A}$  of  $(W_1, \langle, \rangle_{1,F})$ , an element  $v = a_0 + b_0\mathfrak{i} \in \mathbb{SL}_1(F_1)$ , satisfying  $a_0^2 + b_0^2\alpha = 1$ , acts on  $(W_2, \langle, \rangle_{1,F})$  via the following matrix<sup>19</sup>

$$G_v = \begin{pmatrix} a_0 & 0 & \frac{b_0}{2} & 0 \\ 0 & a_0 & 0 & \frac{b_0}{2\beta} \\ -2\alpha b_0 & 0 & a_0 & 0 \\ 0 & -2\alpha\beta b_0 & 0 & a_0 \end{pmatrix}$$

Analogously we have

**Lemma 11.9.** *Let  $B = \begin{pmatrix} 1 & 0 \\ B_{21} & 1 \end{pmatrix} \in \mathrm{GL}_2(M_2(F))$  with  $B_{21} = \begin{pmatrix} 0 & 4\alpha \\ -\beta & 0 \end{pmatrix}$ . Then  $B^{-1}G_vB = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathrm{GL}_2(M_2(F))$ .*

and

**Proposition 11.10.** *Let  $\mathfrak{e}_1 = 1 - \frac{1}{2\alpha}\mathbb{k}$ ,  $\mathfrak{e}_2 = -2\mathfrak{i} + \mathfrak{j}$  be two vectors in  $W_1$ , and  $Y_2 = \mathrm{Span}\{\mathfrak{e}_1, \mathfrak{e}_2\}$ . Then:*

- (1)  $Y_2$  is a Lagrangian subspace of  $(W_1, \langle, \rangle_{1,F})$ .
- (2) The composing mapping

$$\mathbb{SL}_1(F_1) \hookrightarrow \mathrm{U}(W_2, \langle, \rangle_2) \twoheadrightarrow \mathrm{U}(W_2, \langle, \rangle_2) / [\mathrm{U}(W_2, \langle, \rangle_2), \mathrm{U}(W_2, \langle, \rangle_2)]$$

is onto.

- (3) The image  $\iota(\mathbb{SL}_1(F_1))$  in  $\mathrm{Sp}(W_1, \langle, \rangle_{1,F})$  belongs to  $P(Y_2)$ .

**11.4. Cases II & III:**  $\mathfrak{i} = \varpi$  &  $\xi\varpi$ . Suppose now  $\mathfrak{j} = \xi$ , so  $k_{F_2} = k_{\mathbb{H}}$ . For the same reason we have

**Proposition 11.11.** *Let  $Y_2 = \mathrm{Span}\{1, \mathfrak{j}\}$ . Then:*

- (1)  $Y_2$  is a Lagrangian subspace of  $(W_1, \langle, \rangle_{1,F})$ .
- (2) The composing mapping

$$\mathbb{SL}_1(F_2) \hookrightarrow \mathrm{U}(W_2, \langle, \rangle_2) \twoheadrightarrow \mathrm{U}(W_2, \langle, \rangle_2) / [\mathrm{U}(W_2, \langle, \rangle_2), \mathrm{U}(W_2, \langle, \rangle_2)]$$

is onto.

- (3) The image  $\iota(\mathbb{SL}_1(F_2))$  in  $\mathrm{Sp}(W_1, \langle, \rangle_{1,F})$  belongs to  $P(Y_2)$ .

**11.5. The exceptional case III.** Recall that

$$\Gamma = \{(g_1, g_2) \in \mathrm{GU}(W_1) \times \mathrm{GU}(W_2) \mid \lambda_1(g_1)\lambda_2(g_2) = 1\}.$$

Note that, according to Lemma 5.6, the image of  $\Gamma$  in  $\mathrm{Sp}(W_1, \langle, \rangle_{1,F})$  does not totally lie in  $\mathrm{U}(W_1, \langle, \rangle_{1,F_1})$ . Following the same procedure described above about the anisotropic skew hermitian space  $\mathbb{H}(\mathfrak{i})$  we define a smaller subgroup  $\Gamma_1$  of  $\Gamma$ , generated by the elements of the form

$$(I.1) \quad \begin{cases} (i) & (a, a^{-1}), & a \in F^\times, \\ (ii) & (\mathfrak{e}_{-1}\mathfrak{j}, \mathfrak{j}^{-1}), & \text{for some } \mathfrak{e}_{-1} \in F_1^\times, \text{ with } \mathrm{Nrd}(\mathfrak{e}_{-1}) = -1, \\ (iii) & (\mathfrak{j}, \mathfrak{j}^{-1}\mathfrak{e}_{-1}), & \text{above } \mathfrak{e}_{-1}, \end{cases}$$

if  $\mathfrak{i} = \xi$ ,  $-1 \in \mathrm{N}_{F_1/F}(F_1^\times) \setminus (F^\times)^2$ , and

$$\begin{aligned} & \stackrel{19}{(1, \mathfrak{j}; -\frac{1}{2\alpha}\mathfrak{i}, \frac{1}{2\alpha\beta}\mathbb{k})} \cdot (a_0 + b_0\mathfrak{i}) = (a_0 + b_0\mathfrak{i}, a_0\mathfrak{j} - b_0\mathbb{k}, \frac{b_0}{2} - \frac{a_0}{2\alpha}\mathfrak{i}, \frac{b_0}{2\beta}\mathfrak{j} + \frac{a_0}{2\alpha\beta}\mathbb{k}) = (1, \mathfrak{j}; -\frac{1}{2\alpha}\mathfrak{i}, \frac{1}{2\alpha\beta}\mathbb{k}) \cdot \\ & \begin{pmatrix} a_0 & 0 & \frac{b_0}{2} & 0 \\ 0 & a_0 & 0 & \frac{b_0}{2\beta} \\ -2\alpha b_0 & 0 & a_0 & 0 \\ 0 & -2\alpha\beta b_0 & 0 & a_0 \end{pmatrix} \end{aligned}$$



$$(I.2) \begin{cases} (i) & (a, a^{-1}), & a \in F^\times, \\ (ii) & (\mathfrak{e}_{-1}\mathfrak{i}, \mathfrak{i}^{-1}\mathfrak{e}_{-1}), & \text{for some } \mathfrak{e}_{-1} \in F_1^\times, \text{ with } \text{Nrd}(\mathfrak{e}_{-1}) = -1, \\ (iii) & (\mathfrak{j}, \mathfrak{j}^{-1}\mathfrak{e}_{-1}), & \text{above } \mathfrak{e}_{-1}, \end{cases}$$

if  $\mathfrak{i} = \xi$ ,  $-1 \in (F^\times)^2$ , and

$$(II) \begin{cases} (i) & (a, a^{-1}), & a \in F^\times, \\ (ii) & (\mathfrak{i}, \mathfrak{i}^{-1}), \\ (iii) & (\mathfrak{j}, \mathfrak{j}^{-1}\mathfrak{e}_{-1}), & \text{for some } \mathfrak{e}_{-1} \in F_2^\times, \text{ satisfying } \text{Nrd}(\mathfrak{e}_{-1}) = -1, \end{cases}$$

if  $(\mathfrak{i}, \mathfrak{j}) = (\varpi, \xi)$  or  $(\xi\varpi, \xi)$ .

**Lemma 11.12.**  $\Lambda_{\Gamma_1} = F^\times = \Lambda_\Gamma$ .

*Proof.* If  $\mathfrak{i} = \xi$ , then  $\Lambda_{\Gamma_1} = \langle N_{F_1/F}(F_1^\times), -\beta \rangle$  or  $\langle (F_1^\times)^2, \beta, -\beta \rangle$  if  $-1 \in N_{F_1/F}(F_1^\times) \setminus (F^\times)^2$ . Now if  $(\mathfrak{i}, \mathfrak{j}) = (\varpi, \xi)$ , or  $(\xi\varpi, \xi)$ , then by Lemma 4.4,  $-1 \in N_{F_2/F}(F_2^\times)$ , so is  $-\beta$ . Therefore  $N_{F_2/F}(F_2^\times) = (F^\times)^2 \sqcup -\beta(F^\times)^2$ , and consequently  $\Lambda_{\Gamma_1} = \langle N_{F_2/F}(F_2^\times), \alpha \rangle = \langle N_{F_2/F}(F_2^\times), -\alpha \rangle = F^\times$ .  $\square$

**Lemma 11.13.**  $\Gamma_1 \cap (U(W_1, \langle, \rangle_1) \times U(W_2, \langle, \rangle_2)) = \{(-1, -1), (1, 1)\}$ .

*Proof.* Immediately.  $\square$

**Proposition 11.14.** *The condition (C4) of Section 10 holds in above exceptional case.*

*Proof.* 1) In case that  $\mathfrak{i} = \xi$ , and  $k_{F_1} = k_{\mathbb{H}}$ , we choose  $\Omega_\nu = \mathbb{SL}_1(F_1)$ , for  $\nu = 1, 2$ , and  $Y_\nu$  defined in Proposition 11.8 for  $\nu = 1$ , and in Proposition 11.10, for  $\nu = 2$ . As is easily seen that  $\Gamma_1$  stabilizes  $\Omega_\nu$ , so the result holds in this case.

2) In case that  $(\mathfrak{i}, \mathfrak{j}) = (\varpi, \xi)$  or  $(\xi\varpi, \xi)$ , we choose  $\Omega_\nu = \mathbb{SL}_1(F_\nu)$ , for  $\nu = 1, 2$ , and  $Y_\nu$  constructed in Proposition 11.8, for  $\nu = 1$  and in Proposition 11.11, for  $\nu = 2$ . By definition, both  $\Omega_1, \Omega_2$  are  $\Gamma_1$ -stable. This completes the proof.  $\square$

**Proposition 11.15.** *Under the restriction  $H^2(\iota(\Gamma), \mu_8) \longrightarrow H^2(\iota(\Gamma_1), \mu_8)$ , the image of  $[c]$  is trivial.*

*Proof.* *The simple case:*  $-1 \in (F^\times)^2$ . Suppose  $\mathfrak{e}_{-1} \in F^\times$ . The elements  $(\mathfrak{e}_{-1}\mathfrak{i}, \mathfrak{i}^{-1}\mathfrak{e}_{-1})$ ,  $(\mathfrak{i}, \mathfrak{i}^{-1})$  and  $(\mathfrak{j}, \mathfrak{j}^{-1}\mathfrak{e}_{-1})$  act on  $W_1 = F_1 + F_1\mathfrak{j}$  as

$$(\mathfrak{e}_{-1}\mathfrak{i}, \mathfrak{i}^{-1}\mathfrak{e}_{-1}) \cdot (\alpha_0 + \alpha_1\mathfrak{j}) = \mathfrak{e}_{-1}(\alpha_0 - \alpha_1\mathfrak{j})\mathfrak{e}_{-1} \quad (11.1)$$

$$(\mathfrak{i}, \mathfrak{i}^{-1}) \cdot (\alpha_0 + \alpha_1\mathfrak{j}) = (\alpha_0 - \alpha_1\mathfrak{j}) \quad (11.2)$$

$$(\mathfrak{j}, \mathfrak{j}^{-1}\mathfrak{e}_{-1}) \cdot (\alpha_0 + \alpha_1\mathfrak{j}) = (\overline{\alpha_0} + \overline{\alpha_1}\mathfrak{j}) \cdot \mathfrak{e}_{-1} \quad (11.3)$$

$$(\mathfrak{e}_{-1}\mathfrak{j}, \mathfrak{j}^{-1}) \cdot (\alpha_0 + \alpha_1\mathfrak{j}) = \mathfrak{e}_{-1} \cdot (\overline{\alpha_0} + \overline{\alpha_1}\mathfrak{j}) \quad (11.4)$$

for  $\alpha_i \in F_1^\times$ ,  $i = 0, 1$ . Set now  $Y = \{a + d\mathbb{k} | a, d \in F\}$ , which is a Lagrangian subspace of  $(W_1, \langle, \rangle_{1,F})$  because of the equation:  $\langle a + d\mathbb{k}, a' + d'\mathbb{k} \rangle_{1,F} = \text{Trd}((a - d\mathbb{k})\mathfrak{i}(a' + d'\mathbb{k})) = 0$ . By observation  $Y$  is  $\Gamma_1$ -stable, so  $\iota(\Gamma_1)$  lies in  $P(Y)$ .

*The general case I.* Suppose that  $-1$  either lies in  $N_{F_2/F}(F_2^\times)$  or in  $N_{F_3/F}(F_3^\times)$ . We may assume  $\mathfrak{e}_{-1} \in F_2^\times$ . In this situation, the actions of above  $(\mathfrak{e}_{-1}\mathfrak{i}, \mathfrak{i}^{-1}\mathfrak{e}_{-1})$ ,  $(\mathfrak{i}, \mathfrak{i}^{-1})$  and  $(\mathfrak{j}, \mathfrak{j}^{-1}\mathfrak{e}_{-1})$  on  $W_1$  are still given by above equations (11.1), (11.2) and (11.3) respectively. Now we let  $Y = \{a + c\mathfrak{j} | a, c \in F^\times\}$ , which is also a Lagrangian subspace of  $(W_1, \langle, \rangle_{1,F})$ . As  $\mathfrak{e}_{-1} \in F_2^\times$ , it can be checked that the three kinds of elements of  $\Gamma_1$  stabilizes  $Y$ , so  $\iota(\Gamma_1) \subseteq P(Y)$ .

*The general case II.* Suppose that  $-1 \notin (F^\times)^2$  but  $-1 \in N_{F_1/F}(F_1^\times)$ . In this case,  $N_{F_1/F}(F_1^\times) =$

$\langle (F^\times)^2, -1 \rangle$ ,  $F^\times = \langle (F^\times)^2, -1, -\beta \rangle = \langle (F^\times)^2, \beta, -\beta \rangle$  and  $\Gamma_1 = \langle (a, a^{-1}), (\mathfrak{j}, \mathfrak{j}^{-1}e_{-1}), (e_{-1}\mathfrak{j}, \mathfrak{j}^{-1}) \rangle$ , for all  $a \in F^\times$ . By replacing  $\mathfrak{i}$  with certain  $t_0\mathfrak{i}$ , for simplicity, we assume  $\mathfrak{i}^2 = -1$ , and  $e_{-1} = a_{-1} + b_{-1}\mathfrak{i}$ . Let

$$\Gamma_2 = \{(\mu, \nu) \in \Gamma_1 \mid \lambda_1(\mu) = \lambda_2(\nu)^{-1} \in N_{F_2/F}(F_2^\times)\}.$$

It is clear that the image of  $\Gamma_2$  in  $\mathrm{Sp}(W_1, \langle, \rangle_{1,F})$  sits in  $\mathrm{U}(W_1, \langle, \rangle_{1,F_1})$ . Invoking the result of Lemma 11.6, the image  $\overline{\Gamma_2}$  of  $\Gamma_2$  in  $\overline{\mathrm{Sp}(W_1, \langle, \rangle_{1,F})}$  is splitting, and there is an exact sequence

$$1 \longrightarrow \Gamma_2 \longrightarrow \Gamma_1 \longrightarrow F^\times / N_{F_2/F}(F_2^\times) \simeq \mathbb{Z}_2 \longrightarrow 1.$$

We will apply the criterion established in Section 10 to the pair  $(\Gamma_2, \Gamma_1)$  instead of  $(\mathrm{U}(W_1) \times \mathrm{U}(W_2), \Gamma)$ . Under the given symplectic basis  $\mathcal{A}$  of  $\mathrm{Sp}(W_1, \langle, \rangle_{1,F})$ , the image of  $(\mathfrak{j}, \mathfrak{j}^{-1}e_{-1})$  in  $\mathrm{Sp}(W_1, \langle, \rangle_{1,F})$  is represented by the following matrix<sup>20</sup>

$$G_{l_1} = \begin{pmatrix} a_{-1} & 0 & -\frac{b_{-1}}{2} & 0 \\ 0 & a_{-1} & 0 & -\frac{b_{-1}}{2\beta} \\ -2b_{-1} & 0 & -a_{-1} & 0 \\ 0 & -2\beta b_{-1} & 0 & -a_{-1} \end{pmatrix}$$

Now we let  $C = \begin{pmatrix} 1 & 0 \\ C_{21} & 1 \end{pmatrix}$  be an element in  $\mathrm{Sp}(W_1, \langle, \rangle_{1,F})$ , for  $C_{21} = \begin{pmatrix} 2a_{-1}b_{-1}^{-1} & 2b_{-1}^{-1}\beta \\ -2b_{-1}^{-1} & 2a_{-1}b_{-1}^{-1}\beta \end{pmatrix}$ . Then  $C^{-1}G_{l_1}C = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ <sup>21</sup> belongs to  $P(Y)$  for some Lagrangian subspace  $Y$  of  $(W_1, \langle, \rangle_{1,F})$ , provided the condition (C3) (cf. Section 10) of the criterion in our case. We now take

$$\Gamma_2^{(1)} = \langle (e_{-1}\mathfrak{j}, \mathfrak{j}^{-1}) \rangle$$

<sup>20</sup>  $(\mathfrak{j}, \mathfrak{j}^{-1}e_{-1}) \cdot (1, \mathfrak{j}, -\frac{1}{2}\mathfrak{i}, \frac{1}{2\beta}\mathfrak{k}) = (1, \mathfrak{j}, \frac{1}{2}\mathfrak{i}, -\frac{1}{2\beta}\mathfrak{k}) \cdot e_{-1} = (a_{-1} + b_{-1}\mathfrak{i}, a_{-1}\mathfrak{j} - b_{-1}\mathfrak{k}, \frac{a_{-1}}{2}\mathfrak{i} - \frac{b_{-1}}{2}, -\frac{a_{-1}}{2\beta}\mathfrak{k} - \frac{b_{-1}}{2\beta}\mathfrak{j}) = (1, \mathfrak{j}, -\frac{1}{2}\mathfrak{i}, \frac{1}{2\beta}\mathfrak{k})$ .

$$\begin{pmatrix} a_{-1} & 0 & -\frac{b_{-1}}{2} & 0 \\ 0 & a_{-1} & 0 & -\frac{b_{-1}}{2\beta} \\ -2b_{-1} & 0 & -a_{-1} & 0 \\ 0 & -2\beta b_{-1} & 0 & -a_{-1} \end{pmatrix}.$$

<sup>21</sup> Write  $G_{l_1} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ . Then  $C^{-1}G_{l_1}C = \begin{pmatrix} X_{21} - 2C_{21}X_{11} - C_{21}X_{12}C_{21} & * \\ * & * \end{pmatrix}$ . As  $X_{21} - 2C_{21}X_{11} - C_{21}X_{12}C_{21} = -2b_{-1} \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} - 2a_{-1}C_{21} + \frac{b_{-1}}{2}C_{21} \begin{pmatrix} 1 & 0 \\ 0 & \beta^{-1} \end{pmatrix} C_{21}$ , which is zero if  $C_{21} \begin{pmatrix} 1 & 0 \\ 0 & \beta^{-1} \end{pmatrix} C_{21} - 4a_{-1}b_{-1}^{-1}C_{21} = 4 \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}$ , i.e.

$$[C_{21} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2\beta} \end{pmatrix}]^2 - 2a_{-1}b_{-1}^{-1}[C_{21} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2\beta} \end{pmatrix}] = I$$

Let  $A = [C_{21} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2\beta} \end{pmatrix}] = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$ . Then it reduces to solve  $\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} - \begin{pmatrix} 2a_{-1}b_{-1}^{-1} & 0 \\ 0 & 2a_{-1}b_{-1}^{-1} \end{pmatrix} = \begin{pmatrix} \alpha_{22} & -\alpha_{12} \\ -\alpha_{21} & \alpha_{11} \end{pmatrix} \cdot \frac{1}{\det(A)}$ , or  $\begin{cases} \det(A) = -1 \\ \alpha_{11} + \alpha_{22} = 2a_{-1}b_{-1}^{-1} \end{cases}$ . Obviously the elements  $\alpha_{11} = \alpha_{22} = a_{-1}b_{-1}^{-1}$ , and  $\alpha_{12} = b_{-1}^{-1}, \alpha_{21} = -b_{-1}^{-1}$  satisfy the described conditions because of  $a_{-1}^2b_{-1}^{-2} + b_{-1}^{-2} = b_{-1}^{-2}(a_{-1}^2 + 1) = -1$ , so  $C_{21} = \begin{pmatrix} a_{-1}b_{-1}^{-1} & b_{-1}^{-1} \\ -b_{-1}^{-1} & a_{-1}b_{-1}^{-1} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2\beta \end{pmatrix}$ , which is exactly our consideration.

to be a subgroup of  $\Gamma_2$ . Obviously the other two first conditions (C1) and (C2) of the criterion hold for  $\Gamma_2^{(1)}$  and under the given symplectic basis  $\mathcal{A}$  of  $\text{Sp}(W_1, \langle, \rangle_{1,F})$ ,  $(\mathbb{e}_{-1}\mathbb{j}, \mathbb{j}^{-1})$  corresponds to the matrix<sup>22</sup>

$$G_{l_2} = \begin{pmatrix} a_{-1} & 0 & -\frac{b_{-1}}{2} & 0 \\ 0 & a_{-1} & 0 & \frac{b_{-1}}{2\beta} \\ -2b_{-1} & 0 & -a_{-1} & 0 \\ 0 & 2\beta b_{-1} & 0 & -a_{-1} \end{pmatrix}$$

By calculation,  $E^{-1}G_{l_2}E = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$  for  $E = \begin{pmatrix} 1 & 0 \\ E_{21} & 1 \end{pmatrix} \in \text{GL}_4(F)$  with  $E_{21} = \begin{pmatrix} a_{-1}b_{-1}^{-1} & -2b_{-1}^{-1}\beta \\ -b_{-1}^{-1} & -2a_{-1}b_{-1}^{-1}\beta \end{pmatrix} \in \text{GL}_2(F)$ , and  $G_{l_2}G_{l_1}G_{l_2}^{-1} = G_{l_1}$ , so the condition (C4) satisfies. By Lemma 10.1, the extension  $\overline{\Gamma}_1$  is splitting over  $\Gamma_1$ . This completes the whole proof.  $\square$

## 12. THE PROOF OF THE MAIN THEOREM I.

In this section, we follow the notations of Section 10 and prove Theorem A (cf. Section 1) in a more general setting, where either  $W_1$  or  $W_2$  is a hyperbolic space over  $D$ .

**Proposition 12.1.** *When either  $W_1$  or  $W_2$  is a hyperbolic space over  $D$ , Theorem A holds.*

*Proof.* Without loss of generality, we assume that

$$W_2 \simeq n_2 H$$

for hyperbolic plane  $H$  over  $D$ . Let  $H = X \oplus X^*$  be a complete polarisation, so that we can define two sections

$$s^\pm : F^\times \longrightarrow \text{GU}(W_2); a \longmapsto \underbrace{t_a^\pm \times \cdots \times t_a^\pm}_{n_2}$$

for  $t_a^\pm = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm a \end{pmatrix}$ . Let

$$\Gamma_1 = \{(g_1, g_2) \in \text{GU}(W_1) \times s^\pm(F^\times) \mid \lambda_1(g_1) = \lambda_2(g_2)^{-1}\}$$

be a subgroup of  $\Gamma$ . We identify  $\Lambda_\Gamma = \Lambda_{\Gamma_1}$ , and fairly have

$$\Gamma_1 \cap (\text{U}(W_1, \langle, \rangle_1) \times \text{U}(W_2, \langle, \rangle_2)) = \{(1, 1), (-1, -1)\}.$$

As is known that  $\Gamma_1$  belongs to a parabolic subgroup  $P(W_1 \otimes X^*)$  of  $\text{Sp}(W, \langle, \rangle)$ , provided the condition (C3). By Propositions<sup>23</sup> 2.12, 2.15, above parabolic subgroup  $P(W_1 \otimes X^*)$  contains a set of representatives for  $\text{U}(W_2, \langle, \rangle_2)/[\text{U}(W_2, \langle, \rangle_2), \text{U}(W_2, \langle, \rangle_2)]$ , and also  $\text{U}(W_1, \langle, \rangle_1)$ ,  $\Gamma_1$ , so the condition (C4) also holds in this case. By Lemma 10.1, the result follows.  $\square$

**Corollary 12.2.** *If  $W_1$  is a symplectic vector space over  $F$  and  $W_2$  is an orthogonal vector space over  $F$  of even dimension, then  $\overline{\Gamma}$  is splitting over  $\Gamma$ .*

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<sup>22</sup> $(\mathbb{e}_{-1}\mathbb{j}, \mathbb{j}^{-1}) \cdot (1, \mathbb{j}, -\frac{1}{2}\mathbb{i}, \frac{1}{2\beta}\mathbb{k}) = \mathbb{e}_{-1} \cdot (1, \mathbb{j}, \frac{1}{2}\mathbb{i}, -\frac{1}{2\beta}\mathbb{k}) = (a_{-1} + b_{-1}\mathbb{i}, a_{-1}\mathbb{j} + b_{-1}\mathbb{k}, \frac{a_{-1}}{2}\mathbb{i} - \frac{b_{-1}}{2}, -\frac{a_{-1}}{2\beta}\mathbb{k} + \frac{b_{-1}}{2\beta}\mathbb{j}) = (1, \mathbb{j}, -\frac{1}{2}\mathbb{i}, \frac{1}{2\beta}\mathbb{k})$ .

$$\begin{pmatrix} a_{-1} & 0 & -\frac{b_{-1}}{2} & 0 \\ 0 & a_{-1} & 0 & \frac{b_{-1}}{2\beta} \\ -2b_{-1} & 0 & -a_{-1} & 0 \\ 0 & 2\beta b_{-1} & 0 & -a_{-1} \end{pmatrix}$$

<sup>23</sup>Here, we assume that  $W_2$  is not an orthogonal space over  $F$ .

## 13. THE PROOF OF THE MAIN THEOREM II.

In this section we shall prove Theorem A in one type of classes.

**Proposition 13.1.** *Let  $D = E$  be a quadratic field extension of  $F$ , and both  $W_1, W_2$  are anisotropic spaces over  $E$ . Then Theorem A holds.*

*Proof of the proposition:*

**Case I:**  $\dim_E(W_1) = 1$ . We assume  $W_1 = E(f)$ ,  $f = 1$  or  $f \in F^\times \setminus N_{E/F}(E^\times)$ . By definition

$$\langle aw_2, a'w'_2 \rangle_2 = a \langle w_2, w'_2 \rangle_2 \overline{a'} = \overline{aa'} \langle w_2, w'_2 \rangle_2, \quad a, a' \in E^\times, w_2, w'_2 \in W_2.$$

We then define an  $F$ -bilinear map  $\theta$  from  $W \simeq E(f) \otimes_E W_2$  to  $W_2$  as

$$\theta : W = E(f) \otimes_E W_2 \longrightarrow W_2;$$

$$e \otimes w_2 \longmapsto ew_2,$$

which further induces an isometry of symplectic spaces over  $F$ , when  $W_2$  is endowed with the symplectic form  $\langle, \rangle_{2,F} = \text{Tr}_{E/F}(f \langle, \rangle_2)$ . Recall that

$$\Gamma = \{(g_1, g_2) | g_1 \in E^\times, g_2 \in \text{GU}(W_2), \text{ such that } \lambda_1(g_1)\lambda_2(g_2) = 1\}.$$

So the following composite map, induced by above  $\theta$ ,

$$\iota : \Gamma \hookrightarrow \text{Sp}(W, \langle, \rangle) \simeq \text{Sp}(W_2, \langle, \rangle_{2,F}),$$

will imply  $\iota(\Gamma) = \iota(U(W_1, \langle, \rangle_1) \times U(W_2, \langle, \rangle_2))$ . By Theorem 1.1,  $\bar{\Gamma}$  is splitting over  $\Gamma$ .

**Case II:**  $\dim_E(W_2) = 1$ . We assume  $W_2 \simeq (f)E$ , for  $f = 1$  or  $f \in F^\times \setminus N_{E/F}(E^\times)$ , and the skew hermitian form  $\langle, \rangle_2$  on  $W_2$  is given by

$$\langle e_2, e'_2 \rangle_2 = \mu f e_2 \overline{e'_2}, \quad e_2, e'_2 \in E,$$

for some  $\mu \in E^\times$  satisfying  $\bar{\mu}/\mu = -1$ . Similarly as above case we define an  $F$ -bilinear map:

$$\theta : W = W_1 \otimes_E E(f) \longrightarrow W_1; \quad w_1 \otimes e \longmapsto ew_1.$$

Recall that the symplectic form  $\langle, \rangle$  on  $W$  is given by

$$\langle w_1 \otimes e_2, w'_1 \otimes e'_2 \rangle = \text{Tr}_{E/F}(\langle w_1, w'_1 \rangle_1 (-uf \overline{e_2 e'_2})) = \text{Tr}_{E/F}(-uf \langle e_2 w_1, e'_2 w'_1 \rangle_1).$$

Hence above  $\theta$  will induce an isometry from  $(W, \langle, \rangle)$  to  $(W_1, \langle, \rangle_{1,F})$ , when  $\langle, \rangle_{1,F} = \text{Tr}_{E/F}(-\mu f \langle, \rangle_1)$ , and the image of  $\Gamma$  in  $\text{Sp}(W_1, \langle, \rangle_{1,F})$  coincides with that of  $U(W_1, \langle, \rangle_1) \times U(W_2, \langle, \rangle_2)$ . The result then follows.

**Case III:**  $\dim_E(W_1) = \dim_E(W_2) = 2$ . By Theorem 4.1, we assume that  $W_1 \simeq \mathbb{H}$  and  $W_2 \simeq \mathbb{H}$ .

1) Let us formulate explicitly the hermitian form  $\langle, \rangle_1$  and the skew hermitian form  $\langle, \rangle_2$  on  $\mathbb{H}$  (cf. Section 5). Suppose now  $E = F(\mathfrak{i})$  for some pure quaternion  $0 \neq \mathfrak{i} \in \mathbb{H}$ , with  $\mathfrak{i}^2 = -\alpha \in F^\times$ . By [20, Page 358], we can choose an element  $\mathfrak{j}$  of  $\mathbb{H}$ , such that  $\{1, \mathfrak{i}, \mathfrak{j}, \mathfrak{k} = \mathfrak{i}\mathfrak{j}\}$  forms a standard basis of  $\mathbb{H}$ , with  $\mathfrak{i}^2 = -\alpha$  and  $\mathfrak{j}^2 = -\beta$ . Let  $\text{Tr}_{\mathbb{H}/E}$  denote the canonical projection from  $\mathbb{H}$  to  $E$  defined by

$$\text{Tr}_{\mathbb{H}/E}(e_1 + \mathfrak{j}e_2) = e_1, \quad e_i \in E.$$

Then the  $E$ -vector space  $\mathbb{H} = E \oplus \mathfrak{j}E$ , equipped with the form defined as

$$\langle e_1 + \mathfrak{j}e_2, e'_1 + \mathfrak{j}e'_2 \rangle_1 = \text{Tr}_{\mathbb{H}/E}(\overline{(e_1 + \mathfrak{j}e_2)}(e'_1 + \mathfrak{j}e'_2)) = \overline{e_1}e'_1 + \beta \overline{e_2}e'_2,$$

will turn into a hermitian space over  $E$ . On the other hand we assume that  $\langle, \rangle_2$  on  $\mathbb{H} = E \oplus E\mathfrak{j}$  is just given by  $-\mathfrak{i}\langle, \rangle_1$ , i.e., if  $e_1 + e_2\mathfrak{j}, e'_1 + e'_2\mathfrak{j} \in \mathbb{H}$ , we then have

$$\langle e_1 + e_2\mathfrak{j}, e'_1 + e'_2\mathfrak{j} \rangle_2 = -\mathfrak{i}(e_1\overline{e'_1} + \beta e_2\overline{e'_2}) = \text{Tr}_{\mathbb{H}/E} \left( -\mathfrak{i}(e_1 + e_2\mathfrak{j})(\overline{e'_1} + \overline{e'_2\mathfrak{j}}) \right), \quad e_1, e'_1, e_2, e'_2 \in E.$$

2) The symplectic form  $\langle, \rangle$  on  $W = \mathbb{H} \otimes_E \mathbb{H}$  is defined as

$$\begin{aligned} \langle w_1 \otimes w_2, w'_1 \otimes w'_2 \rangle &= \text{Tr}_{E/F} \left( \langle w_1, w'_1 \rangle_1 \overline{\langle w_2, w'_2 \rangle_2} \right) = \text{Tr}_{E/F} \left( \mathfrak{i}(\overline{a_1}a'_1 + \beta\overline{a_2}a'_2)(b'_1\overline{b_1} + \beta b'_2\overline{b_2}) \right) \\ &= \text{Tr}_{E/F} \left( \mathfrak{i}(\overline{a_1}b_1a'_1b'_1 + \beta\overline{a_1}b_2a'_1b'_2) \right) + \text{Tr}_{E/F} \left( \mathfrak{i}\beta(\overline{a_2}b_1a'_2b'_1 + \beta\overline{a_2}b_2a'_2b'_2) \right) \end{aligned}$$

for  $w_1 = a_1 + \mathfrak{j}a_2, w'_1 = a'_1 + \mathfrak{j}a'_2 \in \mathbb{H}; w_2 = b_1 + b_2\mathfrak{j}, w'_2 = b'_1 + b'_2\mathfrak{j} \in \mathbb{H}$ . As is easy to see that there exists an  $E$ -linear map:

$$\theta = \theta_1 \oplus \theta_2 : W = \mathbb{H} \otimes_E \mathbb{H} \simeq (E \oplus \mathfrak{j}E) \otimes_E \mathbb{H} \longrightarrow \mathbb{H} \oplus \mathbb{H};$$

$$[(a_1 + \mathfrak{j}a_2) \otimes (b_1 + b_2\mathfrak{j})] \longmapsto (a_1b_1 + a_1b_2\mathfrak{j}, a_2b_1 + a_2b_2\mathfrak{j}).$$

Then for  $w_1 = a_1 + \mathfrak{j}a_2, w'_1 = a'_1 + \mathfrak{j}a'_2 \in \mathbb{H}; w_2 = b_1 + b_2\mathfrak{j}, w'_2 = b'_1 + b'_2\mathfrak{j} \in \mathbb{H}$ , we get

$$\begin{aligned} \text{Tr}_{E/F} (\overline{\langle \theta_1(w_1 \otimes w_2), \theta_1(w'_1 \otimes w'_2) \rangle_2}) &= \text{Tr}_{E/F} (\overline{\langle a_1b_1 + a_1b_2\mathfrak{j}, a'_1b'_1 + a'_1b'_2\mathfrak{j} \rangle_2}) \\ &= \text{Tr}_{E/F} (-\mathfrak{i}(a_1b_1\overline{a'_1b'_1} + \beta a_1b_2\overline{a'_1b'_2})) = \text{Tr}_{E/F} (\mathfrak{i}(\overline{a_1}b_1a'_1b'_1 + \beta\overline{a_1}b_2a'_1b'_2)) \end{aligned}$$

and

$$\begin{aligned} \text{Tr}_{E/F} (\overline{\langle \theta_2(w_1 \otimes w_2), \theta_2(w'_1 \otimes w'_2) \rangle_2}) &= \text{Tr}_{E/F} (\overline{\langle \beta(a_2b_1 + a_2b_2\mathfrak{j}), a'_2b'_1 + a'_2b'_2\mathfrak{j} \rangle_2}) \\ &= \text{Tr}_{E/F} (-\beta\mathfrak{i}(a_2b_1\overline{a'_2b'_1} + \beta a_2b_2\overline{a'_2b'_2})) = \text{Tr}_{E/F} (\beta\mathfrak{i}(\overline{a_2}b_1a'_2b'_1 + \beta\overline{a_2}b_2a'_2b'_2)). \end{aligned}$$

So we verify that, if  $\mathbb{H} \oplus \mathbb{H}$  is endowed with the symplectic form:

$$\langle, \rangle_{\mathbb{H} \oplus \mathbb{H}} = \text{Tr}_{E/F} (\langle, \rangle_2) + \text{Tr}_{E/F} (\beta\langle, \rangle_2),$$

then the above map  $\theta$  defines an isometry from  $(\mathbb{H} \otimes_E \mathbb{H}, \langle, \rangle)$  to  $(\mathbb{H} \oplus \mathbb{H}, \langle, \rangle_{\mathbb{H} \oplus \mathbb{H}})$ .

3) By definition, we can embed  $E^\times$  into  $\text{GU}(W_1)$  resp.  $\text{GU}(W_2)$  defined as  $[e, e_1 + \mathfrak{j}e_2] := [e_1e + \mathfrak{j}e_2e]$  resp.  $[e, e_1 + e_2\mathfrak{j}] := [ee_1 + ee_2\mathfrak{j}]$ , for  $e \in E^\times$ ; both multipliers of  $e$  are  $N_{E/F}(e)$ . We choose an element  $\mathfrak{e}_{-1} \in \mathbb{H}^\times$ , such that  $\text{Nrd}(\mathfrak{e}_{-1}) = -1$ . Now let  $\mathfrak{e}_{-1}\mathfrak{j}$  act on  $W_1$  as

$$[\mathfrak{e}_{-1}\mathfrak{j}, e_1 + \mathfrak{j}e_2] := \mathfrak{e}_{-1}\mathfrak{j} \cdot (e_1 + \mathfrak{j}e_2),$$

and on  $W_2$  as

$$[\mathfrak{e}_{-1}\mathfrak{j}, e_1 + e_2\mathfrak{j}] := (e_1 + e_2\mathfrak{j}) \cdot \mathfrak{e}_{-1}\mathfrak{j}$$

with the multipliers both being  $\text{Nrd}(\mathfrak{e}_{-1}\mathfrak{j}) = -\beta$ . In this way, we let  $\Gamma_1$  be the subgroup of  $\Gamma$  generated

by  $\begin{cases} (A1) (e, e^{-1}) & \text{for all } e \in E^\times \\ (A2) (\mathfrak{j}, \mathfrak{j}^{-1}) & \text{if } \mathfrak{e}_{-1} \in E^\times \end{cases}$  or by  $\begin{cases} (B1) (e, e^{-1}) & \text{for all } e \in E^\times \\ (B2) (\mathfrak{e}_{-1}\mathfrak{j}, (\mathfrak{e}_{-1}\mathfrak{j})^{-1}) & \text{if } \mathfrak{e}_{-1} \in F(\mathfrak{j})^\times \end{cases}$ .<sup>24</sup> Then:

$$(1) \lambda(\Gamma_1) = \begin{cases} \langle N_{E/F}(E^\times), \beta \rangle = F^\times & \text{if } \mathfrak{e}_{-1} \in E^\times, \\ \langle N_{E/F}(E^\times), -\beta \rangle = F^\times & \text{if } \mathfrak{e}_{-1} \in F(\mathfrak{j})^\times. \end{cases}$$

(2) Recall the map:  $\iota : \Gamma_1 \hookrightarrow \Gamma \longrightarrow \text{Sp}(\mathbb{H} \oplus \mathbb{H}, \langle, \rangle_{\mathbb{H} \oplus \mathbb{H}})$ . By the definition of  $\theta$  we have  $\iota(e) = 1$ , for all  $e \in E^\times$ , and all  $\pm(\mathfrak{j}, \mathfrak{j}^{-1}), \pm(\mathfrak{e}_{-1}\mathfrak{j}, (\mathfrak{e}_{-1}\mathfrak{j})^{-1})$  don't belong to  $\text{U}(W_1, \langle, \rangle_1) \times \text{U}(W_2, \langle, \rangle_2)$ , so  $\iota(\Gamma_1) \cap \iota(\text{U}(W_1, \langle, \rangle_1) \times \text{U}(W_2, \langle, \rangle_2)) = 1$ ;

<sup>24</sup>By Lemma 11.3, without loss of generality, replacing  $\mathfrak{j}$  with  $\mathbb{k}$ , we assume  $\mathfrak{e}_{-1} \in F(\mathfrak{j})^\times$  in this case.

- (3) **Case 1:**  $\mathfrak{e}_{-1} \in E^\times$ . By definition, we have  $\mathfrak{j} \cdot (a + \mathfrak{j}b) = -\beta b + \mathfrak{j}a$  and  $(a + \mathfrak{j}b) \cdot \mathfrak{j}^{-1} = b - \frac{a}{\beta}\mathfrak{j}$ , so through above  $\theta$  the element  $(\mathfrak{j}, \mathfrak{j}^{-1})$  acts on  $\mathbb{H} \oplus \mathbb{H}$  as

$$(\mathfrak{j}, \mathfrak{j}^{-1}) \bullet (a_1 + b_1\mathfrak{j}, a_2 + b_2\mathfrak{j}) = (-\beta b_2 + a_2\mathfrak{j}, b_1 - \frac{a_1}{\beta}\mathfrak{j}).^{25}$$

Now let us choose a symplectic basis  $\mathcal{A}_1 = \{e_1 = \frac{1}{2\alpha}, e_2 = \frac{\mathfrak{j}}{2\alpha\beta}, e_1^* = -\mathfrak{i}, e_2^* = -\mathbb{k}\}$  of  $(\mathbb{H}, \text{Tr}_{E/F}(\overline{\langle, \rangle}_2))$ , and  $\mathcal{A}_2 = \{f_1 = \frac{1}{2\alpha}, f_2 = \frac{\mathfrak{j}}{2\alpha\beta}, f_1^* = -\frac{\mathfrak{i}}{\beta}, f_2^* = -\frac{\mathbb{k}}{\beta}\}$  of  $(\mathbb{H}, \text{Tr}_{E/F}(\beta\overline{\langle, \rangle}_2))$ . Under above basis, the elements  $(\mathfrak{j}, \mathfrak{j}^{-1})$  acts on  $\mathbb{H} \oplus \mathbb{H}$  via the matrix<sup>26</sup>

$$G_{\mathbb{H}, \mathbb{H}} = \begin{bmatrix} \begin{pmatrix} 0 & -1 \\ \beta & 0 \end{pmatrix} & 0 \\ \begin{pmatrix} 0 & \frac{1}{\beta} \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \end{bmatrix}$$

Let  $\mathcal{A} = \{e_1, e_2, f_1, f_2; e_1^*, e_2^*, f_1^*, f_2^*\}$  be a symplectic basis of  $(\mathbb{H} \oplus \mathbb{H}, \langle, \rangle_{\mathbb{H} \oplus \mathbb{H}})$ . Then  $G_{\mathbb{H}, \mathbb{H}}$  should be transferred to a matrix

$$G_{\mathbb{H} \oplus \mathbb{H}} = \begin{bmatrix} \begin{pmatrix} 0 & -1 \\ \beta & 0 \end{pmatrix} & 0 \\ \begin{pmatrix} 0 & \frac{1}{\beta} \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \end{bmatrix}$$

which belongs to the parabolic subgroup  $P(Y)$  for the Lagrangian subspace  $Y = \text{Span}\{e_1^*, e_2^*, f_1^*, f_2^*\}$  of  $W = \mathbb{H} \oplus \mathbb{H}$ . Hence the condition (C3) holds.

**Case 2:**  $\mathfrak{e}_{-1} = a_{-1} + \mathfrak{j}b_{-1} \in F(\mathfrak{j})^\times$ , for some  $a_{-1}, b_{-1} \in F^\times$ . By definition,  $\mathfrak{e}_{-1}\mathfrak{j} \cdot (a + \mathfrak{j}b) =$

<sup>25</sup>  $(\mathfrak{j}, \mathfrak{j}^{-1}) \bullet (a_1 + b_1\mathfrak{j}, a_2 + b_2\mathfrak{j}) = (\mathfrak{j}, \mathfrak{j}^{-1}) \cdot [\theta(1 \otimes (a_1 + b_1\mathfrak{j})) + \theta(\mathfrak{j} \cdot 1 \otimes (a_2 + b_2\mathfrak{j}))] = \theta[\mathfrak{j} \otimes (a_1\mathfrak{j}^{-1} + b_1)] + \theta[(-\beta) \otimes (a_2\mathfrak{j}^{-1} + b_2)] = (-\beta b_2 + a_2\mathfrak{j}, b_1 - \frac{a_1}{\beta}\mathfrak{j})$

<sup>26</sup> Suppose now that  $a_1 = x_1 + x_2\mathfrak{i}, b_1 = x_3 + x_4\mathfrak{i}$ . Then  $a_1 + b_1\mathfrak{j} = x_1 + x_2\mathfrak{i} + x_3\mathfrak{j} + x_4\mathbb{k} = 2\alpha x_1 e_1 - x_2 e_1^* + 2\alpha\beta x_3 e_2 - x_4 e_2^*$ , and  $b_1 - \frac{a_1}{\beta}\mathfrak{j} = -\frac{x_1}{\beta}\mathfrak{j} - \frac{x_2}{\beta}\mathbb{k} + x_3 + x_4\mathfrak{i} = 2\alpha x_3 f_1 - 2\alpha x_1 f_2 - \beta x_4 f_1^* + x_2 f_2^*$ . So  $(\mathfrak{j}, \mathfrak{j}^{-1})$  sends  $e_1$  to  $-f_2$ ,  $e_2$  to  $\frac{1}{\beta}f_1$ ,  $e_1^*$  to

$-f_2^*$ , and  $e_2^*$  to  $\beta f_1^*$  “corresponding to a matrix”  $G_{21} = \begin{pmatrix} \begin{pmatrix} 0 & \frac{1}{\beta} \\ -1 & 0 \end{pmatrix} & 0 \\ \begin{pmatrix} 0 & \beta \\ -1 & 0 \end{pmatrix} & \end{pmatrix}$ . Suppose  $a_2 = x_1 + x_2\mathfrak{i}, b_2 = x_3 + x_4\mathfrak{i}$ . Then

$a_2 + b_2\mathfrak{j} = x_1 + x_2\mathfrak{i} + x_3\mathfrak{j} + x_4\mathbb{k} = 2\alpha x_1 f_1 - \beta x_2 f_1^* + 2\alpha\beta x_3 f_2 - \beta x_4 f_2^*$ , and  $-\beta b_2 + a_2\mathfrak{j} = -\beta(x_3 + x_4\mathfrak{i}) + x_1\mathfrak{j} + x_2\mathbb{k} = -2\alpha\beta x_3 e_1 + 2\alpha\beta x_1 e_2 + x_4\beta e_1^* - x_2 e_2^*$ . The  $(\mathfrak{j}, \mathfrak{j}^{-1})$  sends  $f_1$  to  $\beta e_2$ ,  $f_2$  to  $-e_1$ ,  $f_1^*$  to  $\frac{1}{\beta}e_2^*$ , and  $f_2^*$  to  $-e_1^*$  “corresponding to a

matrix”  $G_{12} = \begin{pmatrix} \begin{pmatrix} 0 & -1 \\ \beta & 0 \end{pmatrix} & 0 \\ \begin{pmatrix} 0 & -1 \\ \frac{1}{\beta} & 0 \end{pmatrix} & \end{pmatrix}$ . Finally,  $(\mathfrak{j}, \mathfrak{j}^{-1}) \cdot (e_1, e_2, e_1^*, e_2^*; f_1, f_2, f_1^*, f_2^*) = (e_1, e_2, e_1^*, e_2^*; f_1, f_2, f_1^*, f_2^*) \cdot \begin{bmatrix} & G_{12} \\ G_{21} & \end{bmatrix}$ .

$-\beta b e_{-1} + a e_{-1} j$ , and  $(a + b j) \cdot (e_{-1} j)^{-1} = (a + b j) \cdot \frac{j}{\beta} \overline{e_{-1}} = -b \overline{e_{-1}} + \frac{a \overline{e_{-1}}}{\beta} j$ , so through above  $\theta$ , the element  $(e_{-1} j, (e_{-1} j)^{-1})$  acts on  $\mathbb{H} \oplus \mathbb{H}$  as

$$\begin{aligned} & (e_{-1} j, (e_{-1} j)^{-1}) \bullet [a_1 + b_1 j, a_2 + b_2 j] \\ &= [(b_1 b_{-1} + b_2 a_{-1})\beta - (a_1 b_{-1} - a_2 a_{-1})j, (-a_{-1} b_1 + \beta b_{-1} b_2) + (\frac{a_1 a_{-1}}{\beta} - a_2 b_{-1})j] \bullet \overline{e_{-1}}.^{27} \end{aligned}$$

By observation, the set  $Y = \{(x_1 + y_1 j), (x_2 + y_2 j) \mid x_i, y_i \in F\}$  is a Lagrangian subspace of  $(\mathbb{H} \oplus \mathbb{H}, \langle, \rangle_{\mathbb{H} \oplus \mathbb{H}})$ , which is also  $(e_{-1} j, (e_{-1} j)^{-1})$ -stable. Hence the condition (C3) holds.

- (4) Let us show the condition (C4) holding in this case. Notations being as in Section 6.2, we have  $[U(W_i, \langle, \rangle_i), U(W_i, \langle, \rangle_i)] \simeq \mathbb{SL}_1(\mathbb{H}) \cap (1 + \mathfrak{P})$ . By almost symmetry, we only deal with the group  $U(W_1, \langle, \rangle_1)$  in the following.

**Case 1:**  $E = F(\xi)$ , with  $\mathfrak{i} = t_0 \xi$  for some  $t_0 \in F^\times$ . By Proposition 6.2, the set

$$\Omega_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{SL}_1(E) \right\}$$

represents the quotient group  $U(W_1, \langle, \rangle_1) / [U(W_1, \langle, \rangle_1), U(W_1, \langle, \rangle_1)]$  in  $U(W_1, \langle, \rangle_1)$ . Note that

$g = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \Omega_1$  maps an element  $\alpha_1 + j\alpha_2 \in W_1$  to  $\alpha_1 a + j\alpha_2 d$ , and in turn via  $\theta$  sends  $[a_1 + b_1 j, a_2 + b_2 j] \in \mathbb{H} \oplus \mathbb{H}$  to  $[a_1 a + b_1 a j, a_2 d + b_2 d j]$ . Similarly, by Proposition 6.2,  $j$  corresponds to the matrix  $\begin{pmatrix} 0 & -\beta \\ 1 & 0 \end{pmatrix} \in \text{GU}(W_1, \langle, \rangle_1)$ , so  $\begin{pmatrix} 0 & -\beta \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -\frac{1}{\beta} & 0 \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix} \in \Omega_1$ , which implies that  $\Omega_1^{\Gamma_1} = \Omega_1$ . Suppose now that  $a = x_1 + x_2 \mathfrak{i}$ ,  $d = x_3 + x_4 \mathfrak{i}$ . Under the given symplectic basis  $\mathcal{A}_1, \mathcal{A}_2$  in above (3) Case 1,  $a$  sends  $\{e_1 = \frac{1}{2\alpha}, e_2 = \frac{1}{2\alpha\beta} j, e_1^* = -\mathfrak{i}, e_2^* = -\mathbb{k}\}$  to  $\{\frac{x_1}{2\alpha} + \frac{x_2}{2\alpha} \mathfrak{i} = x_1 e_1 - \frac{x_2}{2\alpha} e_1^*, \frac{x_1}{2\alpha\beta} j + \frac{x_2}{2\alpha\beta} \mathbb{k} = x_1 e_2 - \frac{x_2}{2\alpha\beta} e_2^*, x_2 \alpha - x_1 \mathfrak{i} = 2\alpha^2 x_2 e_1 + x_1 e_1^*, \alpha x_2 j - x_1 \mathbb{k} = 2\alpha^2 \beta x_2 e_2 + x_1 e_2^*\}$ , which corresponds to the matrix

$$G_{l_1} = \begin{pmatrix} x_1 & 0 & 2\alpha^2 x_2 & 0 \\ 0 & x_1 & 0 & 2\alpha^2 \beta x_2 \\ -\frac{x_2}{2\alpha} & 0 & x_1 & 0 \\ 0 & -\frac{x_2}{2\alpha\beta} & 0 & x_1 \end{pmatrix};$$

$d$  sends  $\{f_1 = \frac{1}{2\alpha}, f_2 = \frac{1}{2\alpha\beta} j, f_1^* = -\frac{1}{\beta} \mathfrak{i}, f_2^* = -\frac{1}{\beta} \mathbb{k}\}$  to  $\{\frac{x_3}{2\alpha} + \frac{x_4}{2\alpha} \mathfrak{i} = x_3 f_1 - \frac{x_4 \beta}{2\alpha} f_1^*, \frac{x_3}{2\alpha\beta} j + \frac{x_4}{2\alpha\beta} \mathbb{k} = x_3 f_2 - \frac{x_4}{2\alpha} f_2^*, \frac{x_4 \alpha}{\beta} - \frac{x_3}{\beta} \mathfrak{i} = \frac{2\alpha^2 x_4}{\beta} f_1 + x_3 f_1^*, \frac{x_4 \alpha}{\beta} j - \frac{x_3}{\beta} \mathbb{k} = 2\alpha^2 x_4 f_2 + x_3 f_2^*\}$  corresponds to the matrix

$$G_{l_2} = \begin{pmatrix} x_3 & 0 & \frac{2\alpha^2 x_4}{\beta} & 0 \\ 0 & x_3 & 0 & 2\alpha^2 x_4 \\ -\frac{x_4 \beta}{2\alpha} & 0 & x_3 & 0 \\ 0 & -\frac{x_4}{2\alpha} & 0 & x_3 \end{pmatrix}.$$

As discussed in Section 11.5, we have

$$A^{-1} G_{l_1} A = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \quad B^{-1} G_{l_2} B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

<sup>27</sup>  $(e_{-1} j, (e_{-1} j)^{-1}) \bullet [a_1 + b_1 j, a_2 + b_2 j] = (e_{-1} j, (e_{-1} j)^{-1}) [\theta(1 \otimes (a_1 + b_1 j)) + \theta(j \cdot 1 \otimes (a_2 + b_2 j))] = \theta(j e_{-1} \otimes (-b_1 \overline{e_{-1}} + \frac{a_1}{\beta} \overline{e_{-1} j})) + \theta((- \beta e_{-1}) \otimes (-b_2 \overline{e_{-1}} + \frac{a_2}{\beta} \overline{e_{-1} j})) = \theta((j a_{-1} - \beta b_{-1}) \otimes (-b_1 + \frac{a_1}{\beta} j) \overline{e_{-1}}) + \theta((- \beta a_{-1} - j \beta b_{-1}) \otimes (-b_2 + \frac{a_2}{\beta} j) \overline{e_{-1}}) = [(\beta b_1 - a_1 j) b_{-1} \overline{e_{-1}}, (-b_1 + \frac{a_1}{\beta} j) a_{-1} \overline{e_{-1}}] + [(b_2 \beta - a_2 j) b_{-1} \overline{e_{-1}}, (\beta b_2 - a_2 j) b_{-1} \overline{e_{-1}}] = [(b_1 b_{-1} + b_2 a_{-1})\beta - (a_1 b_{-1} - a_2 a_{-1})j, (-a_{-1} b_1 + \beta b_{-1} b_2) + (\frac{a_1 a_{-1}}{\beta} - a_2 b_{-1})j] \bullet \overline{e_{-1}}.$

for  $A = \begin{pmatrix} 1 & 0 \\ A_{21} & 1 \end{pmatrix} \in \mathrm{GL}_4(F)$  with  $A_{21} = \begin{pmatrix} 0 & \frac{1}{2\alpha\beta} \\ -\frac{1}{2\alpha^2} & 0 \end{pmatrix}$ , and  $B = \begin{pmatrix} 1 & 0 \\ B_{21} & 1 \end{pmatrix} \in \mathrm{GL}_4(F)$  with  $B_{21} = \begin{pmatrix} 0 & \frac{1}{2\alpha} \\ -\frac{\beta}{2\alpha^2} & 0 \end{pmatrix}$ . Similarly as done in Proposition 11.8, we let  $Y_* = \mathrm{Span} \left\{ \frac{1}{2\alpha} + \frac{1}{2\alpha^2}\mathbb{k}, \frac{1}{2\alpha\beta}\mathbb{j} - \frac{1}{2\alpha\beta}\mathbb{i} \right\}$ , which is a Lagrangian subspace of  $(\mathbb{H}, \mathrm{Tr}_{E/F}(\overline{\langle, \rangle_2}))$  as well as  $(\mathbb{H}, \mathrm{Tr}_{E/F}(\beta\overline{\langle, \rangle_2}))$ . Then  $Y = Y_* \oplus Y_*$  is a well-defined Lagrangian subspace of  $(\mathbb{H} \oplus \mathbb{H}, \langle, \rangle_{\mathbb{H} \oplus \mathbb{H}})$ , provided the result in this case.

**Case 2:** we assume  $\mathbb{i} = \varpi$  or  $\xi\varpi$ , and  $\mathbb{j} = \xi$ . Set  $F_2 = F(\xi)$ . In this case, by Proposition 6.2, the set

$$\Omega_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{S}\mathbb{L}_1(F_2) \right\}$$

should be a set of representatives for  $\mathrm{U}(W_1, \langle, \rangle_1) / [\mathrm{U}(W_1, \langle, \rangle_1), \mathrm{U}(W_1, \langle, \rangle_1)]$ . By Proposition 6.3,  $\mathbb{e}_{-1}\mathbb{j}$  corresponds to the matrix  $\begin{pmatrix} 0 & -\beta\overline{\mathbb{e}_{-1}} \\ \mathbb{e}_{-1} & 0 \end{pmatrix} \in \mathrm{GU}(W_1, \langle, \rangle_1)$ , so  $\begin{pmatrix} 0 & -\beta\overline{\mathbb{e}_{-1}} \\ \mathbb{e}_{-1} & 0 \end{pmatrix}$ .

$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \cdot \begin{pmatrix} 0 & -\overline{\mathbb{e}_{-1}} \\ \frac{1}{\beta}\mathbb{e}_{-1} & 0 \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}$ , which implies that  $\Omega_1^{\Gamma_1} = \Omega_1$ . Clearly, the space  $Y_1 = \{[(x_1 + y_1\mathbb{j}), (x_2 + y_2\mathbb{j})] \mid x_i, y_i \in F\}$  is a well-defined Lagrangian subspace of  $\mathbb{H} \oplus \mathbb{H}$ , and  $\iota(\Omega_1) \subseteq P(Y_1)$ .

By Lemma 10.1, the group  $\overline{\Gamma}$  is splitting over  $\Gamma$ . This completes the proof.

In the following Sections 14–16, in view of the technical results of Satake( Section 8) we shall confirm Theorem A in some quaternionic cases.

#### 14. THE PROOF OF THE MAIN THEOREM III.

In this section we shall follow the notations of Section 10. Let  $(W_1 = \mathbb{H}(\mathbb{j}) \oplus \mathbb{H}(\mathbb{l}), \langle, \rangle_1)$  be a right anisotropic shew hermitian space over  $\mathbb{H}$  of 2 dimension such that

- (1)  $\{1, \mathbb{i}, \mathbb{j}, \mathbb{k} = \mathbb{i}\mathbb{j} = -\mathbb{j}\mathbb{i}\}$  is a standard basis of  $\mathbb{H}$ ;
- (2)  $\mathbb{i}^2 = -\alpha, \mathbb{j}^2 = -\beta, \mathbb{l}^2 = -\mathbb{k}^2 = \alpha\beta$ ;
- (3)  $\mathbb{l} = b_0\mathbb{i} + c_0\mathbb{j} + d_0\mathbb{k}$  for some  $b_0, c_0, d_0 \in F$  satisfying  $b_0^2\alpha + c_0^2\beta + d_0^2\alpha\beta = -\alpha\beta$ .

Let  $(W_2 = \mathbb{H}, \langle, \rangle_2)$  be a left hermitian space over  $\mathbb{H}$  of one dimension with the hermitian form defined by  $\langle \mathbb{d}, \mathbb{d}' \rangle_2 = \mathbb{d}\mathbb{d}'$ , for vectors  $\mathbb{d}, \mathbb{d}' \in \mathbb{H}$ . Let  $(W, \langle, \rangle) = (W_1 \otimes_{\mathbb{H}} W_2, \mathrm{Trd}(\langle, \rangle_1 \otimes \overline{\langle, \rangle_2}))$  be as in Section 10. Let  $\{1, \xi, \varpi, \xi\varpi\}$  be the fixed standard basis of  $\mathbb{H}$  given in Section 4. If  $W_1$  is endowed with the  $F$ -symplectic form  $\langle, \rangle_{1,F} = \mathrm{Trd}(\langle, \rangle_1)$ , then one can check that the canonical mapping

$$\theta : W = W_1 \otimes_{\mathbb{H}} W_2 \simeq (\mathbb{H}(\mathbb{j}) \oplus \mathbb{H}(\mathbb{l})) \longrightarrow W_1 = \mathbb{H}(\mathbb{j}) \oplus \mathbb{H}(\mathbb{l});$$

$$w_1 \otimes \mathbb{d} \longmapsto w_1 \mathbb{d},$$

defines an isometry between  $(W, \langle, \rangle)$  and  $(W_1, \langle, \rangle_{1,F})$ . An element  $g \in \mathrm{U}(W_2, \langle, \rangle_2)$  now acts on  $W_1$  in multiplicity on the right-hand side. In the rest of this section we shall prove the result closely along the different cases described in Corollary 4.5 (2).

**14.1. Cases I & II.** We assume  $(\mathbb{i}, \mathbb{j}, \mathbb{l}) = (\varpi, \xi, \mathbb{e}_{-1}\xi\varpi)$  or  $(\xi\varpi, \xi, \mathbb{e}_{-1}\varpi)$  for some  $\mathbb{e}_{-1} \in F(\xi)^\times$  with  $\mathrm{Nrd}(\mathbb{e}_{-1}) = -1$ , in which cases we may and do assume above  $c_0 = 0$ . Let us fix a symplectic basis  $\mathcal{A}_2 = \left\{ -\frac{1}{2\beta}, \frac{\mathbb{i}}{2\alpha\beta}; \mathbb{j}, \mathbb{k} \right\}$  (so that  $\mathrm{Trd}(-\frac{1}{2\beta}\mathbb{j}\mathbb{j}) = 1 = \mathrm{Trd}(\frac{\mathbb{i}}{2\alpha\beta}\mathbb{j}\mathbb{k})$ ) of the subspace  $(\mathbb{H}(\mathbb{j}), \langle, \rangle_{1,F})$  and a symplectic basis  $\mathcal{A}_3 = \left\{ \frac{1}{2}, -\frac{\mathbb{j}}{2\beta}; \frac{\mathbb{l}}{\alpha\beta}, \frac{\mathbb{i}\mathbb{l}}{\alpha\beta} \right\}$  (so that  $\mathrm{Trd}(\frac{1}{2}\mathbb{l}\frac{\mathbb{l}}{\alpha\beta}) = 1 = \mathrm{Trd}(-\frac{\mathbb{j}}{2\beta}\mathbb{l}\frac{\mathbb{i}\mathbb{l}}{\alpha\beta})$ ) of the subspace  $(\mathbb{H}(\mathbb{l}), \langle, \rangle_{1,F})$ .



**Proposition 14.1.** *For each  $v = 1, 2$ , there is a set of representatives  $\Omega_v$  for  $U(W_v)/[U(W_v), U(W_v)]$ , such that  $\Omega_v$  belongs to a parabolic subgroup  $P(Y_v)$  of  $\mathrm{Sp}(W)$ , for some Lagrangian subspace  $Y_v$  of  $W$ .*

*Proof.* 1) In those cases,  $U(W_2, \langle, \rangle_2) \simeq \mathbb{SL}_1(\mathbb{H})$ . Let  $\mathbb{SL}_1(F(\mathbb{j}))$  denote the set  $\{x \in F(\mathbb{j}) \mid N_{F(\mathbb{j})/F}(x) = 1\}$ . By the arguments of Lemma 5.3, there is a canonical surjective projection from  $\mathbb{SL}_1(F(\mathbb{j}))$  to

$$U(W_2, \langle, \rangle_2)/[U(W_2, \langle, \rangle_2), U(W_2, \langle, \rangle_2)],$$

a cyclic group of  $(q + 1)$  order; we can choose a generator of the latter group with inverse image  $g = x_0 + y_0\mathbb{j}$  in  $\mathbb{SL}_1(F(\mathbb{j}))$ . Under the above basis  $\mathcal{A}_2$  resp.  $\mathcal{A}_3$ ,  $g$  acts on  $\mathbb{H}(\mathbb{j})$  resp.  $\mathbb{H}(\mathbb{I})$  by means of the following matrices

$$G_2 = \begin{pmatrix} x_0 & 0 & 2\beta^2 y_0 & 0 \\ 0 & x_0 & 0 & -2\alpha\beta^2 y_0 \\ -\frac{y_0}{2\beta} & 0 & x_0 & 0 \\ 0 & \frac{y_0}{2\alpha\beta} & 0 & x_0 \end{pmatrix} \text{ resp. } G_3 = \begin{pmatrix} x_0 & y_0 & 0 & 0 \\ -y_0\beta & x_0 & 0 & 0 \\ 0 & 0 & x_0 & y_0\beta \\ 0 & 0 & -y_0 & x_0 \end{pmatrix};$$

it can be checked that  $A^{-1}G_2A = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathrm{GL}_2(M_2(F))$  for  $A = \begin{pmatrix} 1 & 0 \\ A_{21} & 0 \end{pmatrix} \in \mathrm{GL}_2(M_2(F))$  with  $A_{21} = \begin{pmatrix} 0 & -\frac{1}{\alpha} \\ -\frac{1}{4\beta^3} & 0 \end{pmatrix} \in M_2(F)$ . Therefore there is a  $G_2$ -stable Lagrangian subspace  $Y_2^{(1)} = \mathrm{Span}\{\mathfrak{e}_1 = -\frac{1}{2\beta} - \frac{1}{4\beta^3}\mathbb{k}, \mathfrak{e}_2 = \frac{1}{2\alpha\beta}\mathbb{i} - \frac{1}{\alpha}\mathbb{j}\}$  of  $(\mathbb{H}(\mathbb{j}), \langle, \rangle_{1,F})$ , and meanwhile the well-defined Lagrangian subspace  $Y_2^{(2)} = \{x + y\mathbb{j} \mid x, y \in F\}$  of  $(\mathbb{H}(\mathbb{I}), \langle, \rangle_{1,F})$  is  $G_2$ -stable; this gives the result for  $U(W_2, \langle, \rangle_2)/[U(W_2, \langle, \rangle_2), U(W_2, \langle, \rangle_2)]$ .

2) Similarly, according to Proposition 8.13, the image of  $\mathbb{SL}_1(D_{F(\mathbb{i})})/[\mathbb{SL}_1(D_{F(\mathbb{i})}), \mathbb{SL}_1(D_{F(\mathbb{i})})]$  together with  $\{(1, 1), (-\alpha, \mathbb{i}^{-1}), (\beta, \sqrt{-\beta}^{-1}), (-\alpha\beta, \mathbb{i}^{-1}\sqrt{-\beta}^{-1})\}$  in  $U(W_1, \langle, \rangle_1)/[U(W_1, \langle, \rangle_1), U(W_1, \langle, \rangle_1)]$  is full. Now let us first describe the action of  $\{(1, 1), (-\alpha, \mathbb{i}^{-1}), (\beta, \sqrt{-\beta}^{-1}), (-\alpha\beta, \mathbb{i}^{-1}\sqrt{-\beta}^{-1})\}$  on  $W_1$  by following the procedure of Section 8. The elements  $\mathbb{i}^{-1}$  (resp.  $\sqrt{-\beta}^{-1}$ ) corresponds to  $h_1 = \mathrm{diag}(\mathbb{i}^{-1}, \mathbb{i}^{-1}, -\mathbb{i}^{-1}, -\mathbb{i}^{-1})$  (resp.  $h_2 = \mathrm{diag}(\sqrt{-\beta}^{-1}, -\sqrt{-\beta}^{-1}, \sqrt{-\beta}^{-1}, -\sqrt{-\beta}^{-1})$ ) in the algebra  $\mathbb{D}_4$  defined in Theorem 8.8. It is known that

$$h_1^{(2)} = \mathrm{diag}\left(\begin{pmatrix} -\alpha^{-1} & \\ & -\alpha^{-1} \end{pmatrix}, \begin{pmatrix} \alpha^{-1} & \\ & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} \alpha^{-1} & \\ & \alpha^{-1} \end{pmatrix}\right)$$

and

$$h_2^{(2)} = \mathrm{diag}\left(\begin{pmatrix} \beta^{-1} & \\ & \beta^{-1} \end{pmatrix}, \begin{pmatrix} -\beta^{-1} & \\ & -\beta^{-1} \end{pmatrix}, \begin{pmatrix} \beta^{-1} & \\ & \beta^{-1} \end{pmatrix}\right).$$

Therefore the elements  $(-\alpha, \mathbb{i}^{-1})$  and  $(\beta, \sqrt{-\beta}^{-1})$  act on  $W_1 \otimes_F F_1$  as well as  $W_1 \otimes_F K$  via the matrices  $-\alpha P^{-1}h_1^{(2)}P = \mathrm{diag}(1, -1, -1)$  and  $\beta P^{-1}h_2^{(2)}P = \mathrm{diag}(1, -1, 1)$  of  $\mathrm{GL}_3(M_2(F))$  respectively. Observe that the images of  $(-\alpha, \mathbb{i}^{-1})$  and  $(\beta, \sqrt{-\beta}^{-1})$  in  $\mathrm{Sp}(W_1, \langle, \rangle_{1,F})$  belong to the centers of  $\mathrm{Sp}(\mathbb{H}(\mathbb{j}), \langle, \rangle_{1,F}) \times \mathrm{Sp}(\mathbb{H}(\mathbb{I}), \langle, \rangle_{1,F})$ .

Next, by Lemma 5.3,  $\mathbb{SL}_1(D_{F(\mathbb{i})})/[\mathbb{SL}_1(D_{F(\mathbb{i})}), \mathbb{SL}_1(D_{F(\mathbb{i})})]$  is a cyclic group of  $(q + 1)$  order generated by the image of  $\mathbb{SL}_1(F(\sqrt{-\beta})) = \{x \in F(\sqrt{-\beta}) \mid N_{F(\sqrt{-\beta})/F}(x) = 1\}$ . An element  $g \in \mathbb{SL}_1(F(\sqrt{-\beta}))$ , say  $g = a + b\sqrt{-\beta}$  with  $a^2 + b^2\beta = 1$ , corresponds to

$$h = \mathrm{diag}(a + b\sqrt{-\beta}, a - b\sqrt{-\beta}, a + b\sqrt{-\beta}, a - b\sqrt{-\beta})$$

in  $\mathbb{D}_4$ . Clearly  $h^{(2)} = \text{diag}(1, A, 1) \in \text{GL}_3(M_2(K))$  for  $A = \begin{pmatrix} (a+b\sqrt{-\beta})^2 & 0 \\ 0 & (a-b\sqrt{-\beta})^2 \end{pmatrix} \in \text{GL}_2(K)$ ; it acts on  $W_1 \otimes_F F_1$  as well as  $W_1 \otimes_F K$  via the matrix  $P^{-1}h^{(2)}P = \text{diag}(1, B, 1)$  for

$$B = \begin{pmatrix} 1 & \sqrt{-\beta} \\ 1 & -\sqrt{-\beta} \end{pmatrix}^{-1} \begin{pmatrix} (a+b\sqrt{-\beta})^2 & 0 \\ 0 & (a-b\sqrt{-\beta})^2 \end{pmatrix} \begin{pmatrix} 1 & \sqrt{-\beta} \\ 1 & -\sqrt{-\beta} \end{pmatrix} = \begin{pmatrix} a^2 - b^2\beta & -2ab\beta \\ 2ab & a^2 - b^2\beta \end{pmatrix}.$$

In this way we understand  $g$  well as an element of  $\mathbf{SU}_V(F)$ , where  $V = \mathbb{H}(\mathfrak{i}) \oplus W_1$  is given at the beginning of Section 8. As shown in Section 7.1, there is a canonical one-to-one mapping from  $\text{U}(W_1, \langle, \rangle_1)$  to  $\mathbf{SU}_V(F)$ . By Example 7.2 and Lemma 8.2,  $g$  should act on  $W_1 = \mathbb{H}(\mathfrak{j}) \oplus \mathbb{H}(\mathbb{1})$  as an element  $(a^2 - b^2\beta + 2ab\mathfrak{j}, 1)$  of  $\text{U}(\mathbb{H}(\mathfrak{j})) \times \text{U}(\mathbb{H}(\mathbb{1}))$ . Under the basis  $\mathcal{A}_2$ , such element acts on  $(\mathbb{H}(\mathfrak{j}), \langle, \rangle_{1,F})$  via the matrix

$$H_2 = \begin{pmatrix} a^2 - b^2\beta & 0 & 4\beta^2 ab & 0 \\ 0 & a^2 - b^2\beta & 0 & 4\alpha\beta^2 ab \\ -\frac{ab}{\beta} & 0 & a^2 - b^2\beta & 0 \\ 0 & -\frac{ab}{\alpha\beta} & 0 & a^2 - b^2\beta \end{pmatrix};$$

it can be checked that  $B^{-1}H_2B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \text{GL}_2(M_2(F))$  for  $B = \begin{pmatrix} 1 & 0 \\ B_{21} & 1 \end{pmatrix} \in \text{GL}_2(M_2(F))$  with  $B_{21} = \begin{pmatrix} 0 & \frac{1}{\alpha} \\ -\frac{1}{4\beta^3} & 0 \end{pmatrix} \in \text{GL}_2(F)$ . Therefore  $H_2$  belongs to a parabolic subgroup  $P(Y_1^{(1)})$  of  $\text{Sp}(\mathbb{H}(\mathfrak{j}), \langle, \rangle_{1,F})$ , where  $Y_1^{(1)} = \text{Span}\left\{\mathfrak{e}_1 = -\frac{1}{2\beta} - \frac{1}{4\beta^3}\mathbb{k}, \mathfrak{e}_2 = \frac{1}{2\alpha\beta}\mathfrak{i} + \frac{1}{\alpha}\mathfrak{j}\right\}$  is a certain Lagrangian subspace of  $(\mathbb{H}(\mathfrak{j}), \langle, \rangle_{1,F})$ . Then we choose another Lagrangian subspace  $Y_1^{(2)}$  for  $(\mathbb{H}(\mathbb{1}), \langle, \rangle_{1,F})$  so that the Lagrangian subspace  $Y_1 = Y_1^{(1)} \oplus Y_1^{(2)}$  of  $(W_1, \langle, \rangle_{1,F})$  satisfies the required condition for  $\text{U}(W_1, \langle, \rangle_1)/[\text{U}(W_1, \langle, \rangle_1)]$ ,  $\text{U}(W_1, \langle, \rangle_1)$ .  $\square$

Note that for any two elements  $g_1 \in \text{GU}(\mathbb{H}(\mathfrak{j}), \langle, \rangle_{1,F})$  and  $g_2 \in \text{GU}(\mathbb{H}(\mathbb{1}), \langle, \rangle_{1,F})$  with the same similitude factor, the Cartesian product  $(g_1, g_2)$  can be viewed as an element of  $\text{GU}(W_1, \langle, \rangle_1)$ .

For simplicity we assume  $\mathfrak{e}_{-1}^{-1} = d_0 + \frac{b_0}{\beta}\mathfrak{j}$ , and  $\mathbb{1} = \mathfrak{e}_{-1}^{-1}\mathbb{k}$ . Similarly as in Section 11 we let  $\Gamma_1$  be a subgroup of  $\Gamma$  generated by the following elements

$$(I) \begin{cases} (i) & (a, a; a^{-1}) & \text{for all } a \in F^\times \\ (ii) & (\mathfrak{e}_{-1}\mathfrak{j}, \mathfrak{j}; (\mathfrak{e}_{-1}\mathfrak{j})^{-1}) \\ (iii) & (\mathfrak{e}_{-1}\mathbb{1}, \mathbb{1}; \mathbb{1}^{-1}) \end{cases}$$

if  $-1 \in (F^\times)^2$ , or by

$$(II) \begin{cases} (i) & (a, a; a^{-1}) & \text{for all } a \in F^\times \\ (ii) & (\mathbb{1}, \mathfrak{e}_{-1}\mathbb{1}; \mathbb{1}^{-1}\mathfrak{e}_{-1}^{-1}) \\ (iii) & (\mathfrak{e}_{-1}\mathbb{1}, \mathbb{1}; \mathbb{1}^{-1}) \end{cases}$$

if  $-1 \in N_{F(\mathfrak{j})/F}(F(\mathfrak{j})^\times) \setminus (F^\times)^2$ . Immediately we have

**Lemma 14.2.** (i)  $\Lambda_{\Gamma_1} = F^\times = \Lambda_\Gamma$ ;  
(ii)  $\Gamma_1 \cap (\text{U}(W_1, \langle, \rangle_1) \times \text{U}(W_2, \langle, \rangle_2)) = \{(1, 1), (-1, -1)\}$ .

**Lemma 14.3.** Notations being as in Section 10, under the restriction  $\text{H}(\iota(\Gamma), \mu_8) \longrightarrow \text{H}(\iota(\Gamma_1), \mu_8)$ , the image of  $[c]$  is trivial.

*Proof.* Case I:  $(-1) \in (F^\times)^2$ . We assume  $\mathfrak{e}_{-1} \in F^\times$ ; the elements  $(\mathfrak{e}_{-1}\mathfrak{j}, \mathfrak{j}; (\mathfrak{e}_{-1}\mathfrak{j})^{-1})$ , and  $(\mathfrak{e}_{-1}\mathbb{1}, \mathbb{1}; \mathbb{1}^{-1})$  act on  $W_1 = \mathbb{H}(\mathfrak{j}) \oplus \mathbb{H}(\mathbb{1})$  as

$$(\mathfrak{e}_{-1}\mathfrak{j}, \mathfrak{j}; (\mathfrak{e}_{-1}\mathfrak{j})^{-1}) \cdot [(a+b\mathfrak{i}+c\mathfrak{j}+d\mathbb{k}), (a'+b'\mathfrak{i}+c'\mathfrak{j}+d'\mathbb{k})] = [(a-b\mathfrak{i}+c\mathfrak{j}-d\mathbb{k}), \mathfrak{e}_{-1}^{-1}(a'-b'\mathfrak{i}+c'\mathfrak{j}-d'\mathbb{k})]$$

$(e_{-1}\mathbb{I}, \mathbb{I}; \mathbb{I}^{-1}) \cdot [(a + b\mathfrak{i} + c\mathfrak{j} + d\mathfrak{k}), (a' + b'\mathfrak{i} + c'\mathfrak{j} + d'\mathfrak{k})] = [e_{-1}(a - b\mathfrak{i} - c\mathfrak{j} + d\mathfrak{k}), (a' - b'\mathfrak{i} - c'\mathfrak{j} + d'\mathfrak{k})]$   
 Let  $Y = \{a + b\mathfrak{i} \mid a, b \in F\}$ ; by observation  $Y$  is a Lagrangian subspace of  $(\mathbb{H}(\mathfrak{j}), \langle, \rangle_{1,F})$  as well as  $(\mathbb{H}(\mathbb{I}), \langle, \rangle_{1,F})$ . Therefore  $\iota(\Gamma_1) \subseteq P(Y \oplus Y)$ , which is a parabolic subgroup of  $\mathrm{Sp}(W_1, \langle, \rangle_{1,F})$ .

Case II. Suppose  $-1 \in N_{F(\mathfrak{j})/F}(F(\mathfrak{j})^\times) \setminus (F^\times)^2$ . By definition,  $\iota(\Gamma_1)$  belongs to  $\mathrm{Sp}(\mathbb{H}(\mathfrak{j}), \langle, \rangle_{1,F}) \times \mathrm{Sp}(\mathbb{H}(\mathbb{I}), \langle, \rangle_{1,F})$ ; we denote its image in the first group by  $\iota(\Gamma_1)^{(1)}$ , and the second one by  $\iota(\Gamma_1)^{(2)}$ . According to the proof of the general case of Proposition 11.15, the covering group  $\iota(\Gamma_1)^{(1)} (\subseteq \mathrm{Sp}(\mathbb{H}(\mathfrak{j}), \langle, \rangle_{1,F}))$  is splitting. On the other hand, we choose a Lagrangian subspace  $Y = \{a + b\mathfrak{j} \mid a, b \in F\}$  of  $\mathbb{H}(\mathbb{I})$ . Then  $\iota(\Gamma_1)^{(2)}$  belongs to a parabolic subgroup  $P(Y)$  of  $\mathrm{Sp}(\mathbb{H}(\mathbb{I}), \langle, \rangle_{1,F})$ , so  $\iota(\Gamma_1)^{(2)}$  is also splitting over  $\iota(\Gamma_1)^{(2)}$ . As a consequence, the original covering group  $\iota(\Gamma_1)$  is splitting.  $\square$

Following the proof of Proposition 14.1, and notations being as in Section 10, we let  $\Omega_1 = \mathbb{SL}_1(F(\sqrt{-\beta}))$ , and  $\Omega_2 = \mathbb{SL}_1(F(\mathfrak{j}))$ .

**Lemma 14.4.** *The condition (C4) of Section 10 holds in above two cases I and II.*

*Proof.* By definition, for  $a_0 + b_0\mathfrak{j} \in \Omega_2$  we have

$$\begin{aligned} (e_{-1}\mathfrak{j})^{-1} \cdot (a_0 + b_0\mathfrak{j}) \cdot (e_{-1}\mathfrak{j}) &\in \Omega_2, \\ (\mathbb{I}^{-1}e_{-1}^{-1}) \cdot (a_0 + b_0\mathfrak{j}) \cdot (e_{-1}\mathbb{I}) &= a_0 - b_0\mathfrak{j} \in \Omega_2, \\ \mathbb{I}^{-1}(a_0 + b_0\mathfrak{j})\mathbb{I} &= a_0 - b_0\mathfrak{j} \in \Omega_2; \end{aligned}$$

so  $\Omega_2^{\Gamma_1} = \Omega_2$ . On the other hand, the process of the proving Proposition 14.1 shows that an element  $g = a + b\sqrt{-\beta} \in \Omega_1 = \mathbb{SL}_1(F(\sqrt{-\beta}))$ , should correspond to an element  $(a^2 - b^2\beta + 2ab\mathfrak{j}, 1) \in \mathrm{U}(\mathbb{H}(\mathfrak{j}), \langle, \rangle_1) \times \mathrm{U}(\mathbb{H}(\mathbb{I}), \langle, \rangle_1)$ . Therefore  $\iota(\Omega_1)^{(\Gamma_1)} = \iota(\Omega_1)$ .  $\square$

By Lemma 10.1, we obtain

**Proposition 14.5.** *In above two cases I and II, Theorem A holds.*

14.2. **Case III.** In this subsection, we shall use the notion and conventions of Section 8.1 freely. Let  $(\mathfrak{i}, \mathfrak{j}, \mathbb{I}) = (\xi, \varpi, e_{-1}\xi\varpi)$ , for  $e_{-1} \in F(\xi)^\times$  with  $N_{F(\xi)/F}(e_{-1}) = -1$  so that we assume  $b_0 = 0$  at the beginning. In this case, we fix the symplectic basis  $\mathcal{B}_2 = \{-\frac{1}{2\beta}, \frac{\mathfrak{i}}{2\alpha\beta}; \mathfrak{j}, \mathfrak{k}\}$  and  $\mathcal{B}_3 = \{\frac{1}{2}, -\frac{\mathfrak{i}}{2\alpha}; \frac{1}{\alpha\beta}, \frac{\mathbb{I}}{\alpha\beta}\}$  of  $(\mathbb{H}(\mathfrak{j}), \langle, \rangle_{1,F})$  and of  $(\mathbb{H}(\mathbb{I}), \langle, \rangle_{1,F})$  respectively.

Now  $\mathrm{U}(W_2, \langle, \rangle_2) \simeq \mathbb{SL}_1(\mathbb{H})$ . Set  $\mathbb{SL}_1(F(\mathfrak{i})) = \{x \in F(\mathfrak{i}) \mid \mathrm{Nrd}(x) = 1\}$ . As is known that

$$\mathrm{U}(W_2, \langle, \rangle_2) / [\mathrm{U}(W_2, \langle, \rangle_2), \mathrm{U}(W_2, \langle, \rangle_2)] \simeq \mathbb{SL}_1(\mathbb{H}) / [\mathbb{SL}_1(\mathbb{H}), \mathbb{SL}_1(\mathbb{H})],$$

is a cyclic group of  $(q + 1)$  order; we choose a generator with inverse image  $x_0 + y_0\mathfrak{i}$  in  $\mathbb{SL}_1(F(\mathfrak{i}))$ . Such element should act on  $\mathbb{H}(\mathfrak{j})$  resp.  $\mathbb{H}(\mathbb{I})$ , with respect to the basis  $\mathcal{B}_2$  resp.  $\mathcal{B}_3$ , via the matrices

$$L_2 = \begin{pmatrix} x_0 & y_0 & 0 & 0 \\ -y_0\alpha & x_0 & 0 & 0 \\ 0 & 0 & x_0 & y_0\alpha \\ 0 & 0 & -y_0 & x_0 \end{pmatrix} \text{ resp. } L_3 = L_2. \text{ We let } Y_\nu = \{x + y\mathfrak{i} \mid x, y \in F\} \text{ be a Lagrangian subspace}$$

of  $(\mathbb{H}(\mathfrak{j}), \langle, \rangle_{1,F})$  as well as  $(\mathbb{H}(\mathbb{I}), \langle, \rangle_{1,F})$ , and  $Y = Y_\nu \oplus Y_\nu$  a Lagrangian subspace of  $(W_1, \langle, \rangle_{1,F})$ . Clearly we have

**Lemma 14.6.** *Let  $\Omega_2 = \mathbb{SL}_1(F(\mathfrak{i}))$ , with full image in  $\mathrm{U}(W_2, \langle, \rangle_2) / [\mathrm{U}(W_2, \langle, \rangle_2), \mathrm{U}(W_2, \langle, \rangle_2)]$ . Then the image of  $\Omega_2$  in  $\mathrm{Sp}(W_1, \langle, \rangle_{1,F})$  belongs to  $P(Y)$ .*

*Proof.* Clearly both  $L_2, L_3$  belongs to the parabolic subgroup  $P(Y)$  of  $\mathrm{Sp}(W_1, \langle, \rangle_{1,F})$ , so we prove the result for  $\mathrm{U}(W_2) / [\mathrm{U}(W_2, \langle, \rangle_2), \mathrm{U}(W_2, \langle, \rangle_2)]$ .  $\square$

Now let us turn to the group  $U(W_1, \langle, \rangle_1)$ . Similarly as above Cases *I* & *II*, the images of  $(-\alpha, \mathfrak{i}^{-1})$  and  $(\beta, \sqrt{-\beta}^{-1})$  in  $\mathrm{Sp}(W_1, \langle, \rangle_{1,F})$  belong to the centers of  $\mathrm{Sp}(\mathbb{H}(\mathfrak{j}), \langle, \rangle_{1,F}) \times \mathrm{Sp}(\mathbb{H}(\mathbb{I}), \langle, \rangle_{1,F})$ . Now let us consider  $\mathbb{S}\mathbb{L}_1(D_{F(\mathfrak{i})})/[\mathbb{S}\mathbb{L}_1(D_{F(\mathfrak{i})}), \mathbb{S}\mathbb{L}_1(D_{F(\mathfrak{i})})]$ , which is a cyclic group of  $(q^2 + 1)$  order. Let  $\alpha_1 + A_1\alpha_2 \in \mathbb{S}\mathbb{L}_1(D_{F(\mathfrak{i})})$  such that  $\alpha_1^2 + (c_0 - d_0\mathfrak{i})\alpha_2^2 = 1$  for some  $\alpha_1, \alpha_2 \in F(\mathfrak{i})$ . Suppose that its image in  $\mathbb{S}\mathbb{L}_1(D_{F(\mathfrak{i})})/[\mathbb{S}\mathbb{L}_1(D_{F(\mathfrak{i})}), \mathbb{S}\mathbb{L}_1(D_{F(\mathfrak{i})})]$  is a generator. Then it acts on  $W_1 \otimes_F F_1$  via the following matrix

$$g = \begin{pmatrix} \alpha_1 & (-c_0 + d_0\mathfrak{i})\alpha_2 & & \\ \alpha_2 & \alpha_1 & & \\ & & \alpha_1^\sigma & -\frac{(c_0+d_0\mathfrak{i})}{\mathfrak{i}}\alpha_2^\sigma \\ & & \mathfrak{i}\alpha_2^\sigma & \alpha_1^\sigma \end{pmatrix} \in \mathbb{D}_4.$$

By calculation (cf. Remark 7.4), we know that

$$g^{(2)} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & \mathbb{N}_{F(\mathfrak{i})/F}(\alpha_1) & \mathfrak{i}\mathbb{N}_{F(\mathfrak{i})/F}(\alpha_2) & -\frac{c_0+d_0\mathfrak{i}}{\mathfrak{i}}\alpha_1\alpha_2^\sigma & (-c_0 + d_0\mathfrak{i})\alpha_2\alpha_1^\sigma \\ & \mathfrak{i}\mathbb{N}_{F(\mathfrak{i})/F}(\alpha_2) & \mathbb{N}_{F(\mathfrak{i})/F}(\alpha_1) & \alpha_2\alpha_1^\sigma & \mathfrak{i}\alpha_1\alpha_2^\sigma \\ & \mathfrak{i}\alpha_1\alpha_2^\sigma & (-c_0 + d_0\mathfrak{i})\alpha_2\alpha_1^\sigma & \mathbb{N}_{F(\mathfrak{i})/F}(\alpha_1) & \mathfrak{i}(-c_0 + d_0\mathfrak{i})\mathbb{N}_{F(\mathfrak{i})/F}(\alpha_2) \\ & \alpha_2\alpha_1^\sigma & -\frac{c_0+d_0\mathfrak{i}}{\mathfrak{i}}\alpha_1\alpha_2^\sigma & -\frac{c_0+d_0\mathfrak{i}}{\mathfrak{i}}\mathbb{N}_{F(\mathfrak{i})/F}(\alpha_2) & \mathbb{N}_{F(\mathfrak{i})/F}(\alpha_1) \end{pmatrix}.$$

Note that  $P = \mathrm{diag}\left(\begin{pmatrix} 1 & 0 \\ 0 & 2\mathfrak{i} \end{pmatrix}, \begin{pmatrix} 1 & \sqrt{-\beta} \\ 1 & -\sqrt{-\beta} \end{pmatrix}, \begin{pmatrix} -c_0 + d_0\mathfrak{i} & \sqrt{-\beta}\mathfrak{i} \\ 1 & \frac{\sqrt{-\beta}\mathfrak{i}}{c_0-d_0} \end{pmatrix}\right)$ . So it can be checked that  $P^{-1}g^{(2)}P =$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & X_{22} & X_{23} \\ 0 & X_{32} & X_{33} \end{pmatrix} \in \mathrm{GL}_3(M_2(F)) \text{ for } X_{22} = \begin{pmatrix} \mathbb{N}_{F(\mathfrak{i})/F}(\alpha_1) + \mathfrak{i}\mathbb{N}_{F(\mathfrak{i})/F}(\alpha_2) & 0 \\ 0 & \mathbb{N}_{F(\mathfrak{i})/F}(\alpha_1) - \mathfrak{i}\mathbb{N}_{F(\mathfrak{i})/F}(\alpha_2) \end{pmatrix},$$

$$X_{23} = \begin{pmatrix} (-c_0 + d_0\mathfrak{i})\alpha_2\alpha_1^\sigma + \alpha_1\alpha_2^\sigma\mathfrak{i} & 0 \\ 0 & -\alpha_2\alpha_1^\sigma\mathfrak{i} - (c_0 + d_0\mathfrak{i})\alpha_1\alpha_2^\sigma \end{pmatrix}, X_{32} = \begin{pmatrix} \alpha_2\alpha_1^\sigma - \frac{(c_0+d_0\mathfrak{i})\alpha_1\alpha_2^\sigma}{\mathfrak{i}} & 0 \\ 0 & \alpha_1\alpha_2^\sigma + \frac{(c_0-d_0\mathfrak{i})\alpha_2\alpha_1^\sigma}{\mathfrak{i}} \end{pmatrix},$$

$$X_{33} = \begin{pmatrix} \mathbb{N}_{F(\mathfrak{i})/F}(\alpha_1) + \frac{(c_0^2-d_0^2\alpha)\mathbb{N}_{F(\mathfrak{i})/F}(\alpha_2)}{\mathfrak{i}} & 0 \\ 0 & \mathbb{N}_{F(\mathfrak{i})/F}(\alpha_1) - \frac{(c_0^2-d_0^2\alpha)\mathbb{N}_{F(\mathfrak{i})/F}(\alpha_2)}{\mathfrak{i}} \end{pmatrix}. \text{ By Lemma 8.2, the matrix}$$

$$\begin{pmatrix} X_{22} & X_{23} \\ X_{32} & X_{33} \end{pmatrix} \text{ corresponds to } \begin{pmatrix} \mathbb{N}_{F(\mathfrak{i})/F}(\alpha_1) + \mathfrak{i}\mathbb{N}_{F(\mathfrak{i})/F}(\alpha_2) & (-c_0 + d_0\mathfrak{i})\alpha_2\alpha_1^\sigma + \alpha_1\alpha_2^\sigma\mathfrak{i} \\ \alpha_2\alpha_1^\sigma - \frac{(c_0+d_0\mathfrak{i})\alpha_1\alpha_2^\sigma}{\mathfrak{i}} & \mathbb{N}_{F(\mathfrak{i})/F}(\alpha_1) + \frac{(c_0^2-d_0^2\alpha)\mathbb{N}_{F(\mathfrak{i})/F}(\alpha_2)}{\mathfrak{i}} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{H}).$$

Taking the set  $\Xi_1$  defined in Proposition 8.13 to be  $\mathbb{S}\mathbb{L}_1(F(\mathfrak{i})(A_1))$ , we then have

**Lemma 14.7.** *Let  $\Omega_1$  be the image of the set  $T_1$  (Proposition 8.13) in  $U(W_1, \langle, \rangle_1)$ . Then*

- (1) *The composite map  $\Omega_1 \hookrightarrow U(W_1, \langle, \rangle_1) \twoheadrightarrow U(W_1, \langle, \rangle_{1,F})/[\mathrm{U}(W_1, \langle, \rangle_{1,F}), \mathrm{U}(W_1, \langle, \rangle_{1,F})]$  is onto.*
- (2) *The image of  $\Omega_1$  in  $\mathrm{Sp}(W_1, \langle, \rangle_{1,F})$  belongs to certain  $P(Y)$ .*

*Proof.* The first statement is immediate. Now recall that  $Y_\nu = \{x + y\mathfrak{i} \mid x, y \in F\}$  is a Lagrangian subspace of  $(\mathbb{H}(\mathfrak{j}), \langle, \rangle_{1,F})$  as well as  $(\mathbb{H}(\mathbb{I}), \langle, \rangle_{1,F})$ . From above, we know that the Lagrangian subspace  $Y = Y_\nu \oplus Y_\nu$  of  $(W_1, \langle, \rangle_{1,F})$  is  $(\alpha_1 + A_1\alpha_2)$ -stable, for  $\alpha_1 + A_1\alpha_2 \in \mathbb{S}\mathbb{L}_1(F(\mathfrak{i})(A_1))$ , which is the required result.  $\square$

We assume  $\mathfrak{e}_{-1}^{-1} = d_0 - \frac{c_0}{\alpha}\mathfrak{i}$ , and  $\mathbb{I} = \mathfrak{e}_{-1}^{-1}\mathbb{k}$ . Now we let  $\Gamma_1$  be a subgroup of  $\Gamma$  generated by the following elements: (i)  $(a, a; a^{-1})$  for all  $a \in F^\times$ , (ii)  $(\mathfrak{i}, \mathfrak{i}; \mathfrak{e}_{-1}\mathfrak{i})$ , (iii)  $(\mathfrak{e}_{-1}\mathbb{I}, \mathbb{I}; \mathbb{I}^{-1})$ . Similarly we have

**Lemma 14.8.** (1)  $\Lambda_{\Gamma_1} = F^\times = \Lambda_\Gamma$ .

- (2)  $\Gamma_1 \cap (\mathrm{U}(W_1, \langle, \rangle_1) \times \mathrm{U}(W_1, \langle, \rangle_1)) = \{(1, 1), (-1, -1)\}.$

**Lemma 14.9.** *The condition (C4) of Section 10 also holds in this case.*

*Proof.* The result follows from the defined Lagrangian subspace  $Y = \{[(x_1 + y_1\mathfrak{i}), (x_2 + y_2\mathfrak{i})] \mid x_i, y_i \in F\}$  for  $\Omega_1$  as well as  $\Omega_2$  in Lemmas 14.7, 14.6 respectively.  $\square$

**Lemma 14.10.** *Notations being as in Section 10, under the restriction  $H^2(\iota(\Gamma), \mu_8) \longrightarrow H^2(\iota(\Gamma_1), \mu_8)$ , the image of  $[c]$  is trivial.*

*Proof.* It is clear that  $\iota(\Gamma_1)$  belongs to  $P(Y)$  in  $\mathrm{Sp}(W_1, \langle, \rangle_{1,F})$ , for above Lagrangian subspace  $Y$  of  $W_1$ .  $\square$

Finally we achieve the main result of this subsection:

**Proposition 14.11.** *In above Case III, Theorem A holds.*

## 15. THE PROOF OF THE MAIN THEOREM IV.

Let  $(W_1 = \mathbb{H}(\mathfrak{i}) \oplus \mathbb{H}(\mathfrak{j}) \oplus \mathbb{H}(\mathfrak{l}), \langle, \rangle_1)$  be a right anisotropic shew hermitian space over  $\mathbb{H}$  of dimension 3 given at the beginning of Section 14. Let  $(W_2 = \mathbb{H}, \langle, \rangle_2)$  be a left hermitian space over  $\mathbb{H}$  of dimension 1. Let  $\{1, \xi, \varpi, \xi\varpi\}$  be the fixed standard basis of  $\mathbb{H}$  in Section 4. By Corollary 4.5, we assume that  $(\mathfrak{i}, \mathfrak{j}, \mathfrak{l}) = (\xi, \varpi, \mathfrak{e}_{-1}\xi\varpi)$  for  $\mathfrak{e}_{-1} \in F(\xi)$ ,  $N_{F(\xi)/F}(\mathfrak{e}_{-1}) = -1$  and  $b_0 = 0$ .

As before we endow  $W_1$  with the  $F$ -symplectic form  $\langle, \rangle_{1,F} = \mathrm{Trd}(\langle, \rangle_1)$  so that the canonical mapping

$$\begin{aligned} \theta : W = W_1 \otimes_{\mathbb{H}} W_2 &\simeq (\mathbb{H}(\mathfrak{i}) \oplus \mathbb{H}(\mathfrak{j}) \oplus \mathbb{H}(\mathfrak{l})) \otimes_{\mathbb{H}} \mathbb{H} \longrightarrow \mathbb{H}(\mathfrak{i}) \oplus \mathbb{H}(\mathfrak{j}) \oplus \mathbb{H}(\mathfrak{l}); \\ w_1 \otimes \mathfrak{d} &\longmapsto w_1 \mathfrak{d} \end{aligned}$$

defines an isometry between  $(W, \langle, \rangle)$  and  $(W_1, \langle, \rangle_{1,F})$ . We fix a basis  $\mathcal{B}_1 = \{-\frac{1}{2\alpha}, \frac{\mathfrak{i}}{2\alpha\beta}; \mathfrak{i}, -\mathfrak{k}\}$  of  $(\mathbb{H}(\mathfrak{i}), \langle, \rangle_{1,F})$ , resp.  $\mathcal{B}_2 = \{-\frac{1}{2\beta}, \frac{\mathfrak{i}}{2\alpha\beta}; \mathfrak{j}, \mathfrak{k}\}$  of  $(\mathbb{H}(\mathfrak{j}), \langle, \rangle_{1,F})$ , resp.  $\mathcal{B}_3 = \{\frac{1}{2}, -\frac{\mathfrak{i}}{2\alpha}; \frac{\mathfrak{l}}{\alpha\beta}, \frac{\mathfrak{il}}{\alpha\beta}\}$  of  $(\mathbb{H}(\mathfrak{l}), \langle, \rangle_{1,F})$ .

In this case  $U(W_2, \langle, \rangle_2) \simeq \mathbb{SL}_1(\mathbb{H})$ . According to the discussion in Section 14.2, the element  $g = x_0 + y_0\mathfrak{i} \in \mathbb{SL}_1(F(\mathfrak{i}))$  acts on  $\mathbb{H}(\mathfrak{i})$ ,  $\mathbb{H}(\mathfrak{j})$ ,  $\mathbb{H}(\mathfrak{l})$ , with respect to above  $\mathcal{B}_1$ ,  $\mathcal{B}_2$ ,  $\mathcal{B}_3$ , by the matrices

$$L_1 = \begin{pmatrix} x_0 & 0 & 2\alpha^2 y_0 & 0 \\ 0 & x_0 & 0 & -2\alpha^2 \beta y_0 \\ -\frac{y_0}{2\alpha} & 0 & x_0 & 0 \\ 0 & \frac{y_0}{2\alpha\beta} & 0 & x_0 \end{pmatrix}, L_2, L_3 \text{ respectively, where } L_2, L_3 \text{ are the matrices given there.}$$

We firstly have

**Lemma 15.1.** *Let  $N = \begin{pmatrix} 1 & 0 \\ N_{21} & 1 \end{pmatrix} \in \mathrm{GL}_2(M_2(F))$  with  $N_{21} = \begin{pmatrix} 0 & \frac{1}{2\alpha\beta} \\ \frac{1}{2\alpha^2} & 0 \end{pmatrix}$ . Then  $N^{-1}L_1N = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathrm{GL}_2(M_2(F))$ .*

*Proof.*  $N^{-1}L_1N = \begin{pmatrix} 1 & 0 \\ -N_{21} & 1 \end{pmatrix} \cdot \begin{pmatrix} x_0 \cdot 1 & 2\alpha^2 y_0 \mathrm{diag}(1, -\beta) \\ -\frac{y_0}{2\alpha} \mathrm{diag}(1, -\beta^{-1}) & x_0 \cdot 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ N_{21} & 1 \end{pmatrix} = \begin{pmatrix} * & * \\ \mathcal{X} & * \end{pmatrix}$ , for  $\mathcal{X} = -\frac{y_0}{2\alpha} \begin{pmatrix} 1 & 0 \\ 0 & -\beta^{-1} \end{pmatrix} - 2\alpha^2 y_0 N_{21} \begin{pmatrix} 1 & 0 \\ 0 & -\beta \end{pmatrix} N_{21}$ . As  $N_{21} \begin{pmatrix} 1 & 0 \\ 0 & -\beta \end{pmatrix} N_{21} = -\frac{1}{4\alpha^3} \begin{pmatrix} 1 & 0 \\ 0 & -\beta^{-1} \end{pmatrix}$ , which gives the result.  $\square$

and

**Proposition 15.2.** *Let  $\mathfrak{e}_1 = -\frac{1}{2\alpha} - \frac{1}{2\alpha^2}\mathfrak{k}$ ,  $\mathfrak{e}_2 = \frac{1}{2\alpha\beta}\mathfrak{j} + \frac{1}{2\alpha\beta}\mathfrak{i}$  be two vectors in  $\mathbb{H}(\mathfrak{i})$ , and  $Y_1 = \mathrm{Span}\{\mathfrak{e}_1, \mathfrak{e}_2\}$ . Then:*

- (1)  $Y_1$  is a Lagrangian subspace of  $(\mathbb{H}(\mathfrak{i}), \langle, \rangle_{1,F})$ .
- (2) The image of  $\mathbb{SL}_1(F(\mathfrak{i}))$  in  $\mathrm{Sp}(\mathbb{H}(\mathfrak{i}), \langle, \rangle_{1,F})$  belongs to  $P(Y_1)$ .

We now let  $Y_* = \{x + y\mathfrak{i} \mid x, y \in F\}$  be a Lagrangian subspace of  $(\mathbb{H}(\mathfrak{j}), \langle, \rangle_{1,F})$  as well as  $(\mathbb{H}(\mathfrak{l}), \langle, \rangle_{1,F})$ , and  $Y = Y_1 \oplus Y_* \oplus Y_*$  a Lagrangian subspace of  $(W_1, \langle, \rangle_{1,F})$ . We have

**Lemma 15.3.** *Let  $\Omega_2 = \mathbb{SL}_1(F(\mathfrak{i}))$ , with full image in  $U(W_2, \langle, \rangle_2)/[U(W_2, \langle, \rangle_2), U(W_2, \langle, \rangle_2)]$ . Then the image of  $\Omega_2$  in  $\text{Sp}(W_1, \langle, \rangle_{1,F})$  belongs to  $P(Y)$ .*

Now let us consider  $U(W_1, \langle, \rangle_1)$ . By Proposition 8.10, the group  $U(W_1, \langle, \rangle_1)/[U(W_1, \langle, \rangle_1), U(W_1, \langle, \rangle_1)]$  is generated by the images of  $\mathbb{SL}_1(\mathbb{D}_4)/[\mathbb{SL}_1(\mathbb{D}_4), \mathbb{SL}_1(\mathbb{D}_4)]$  and

$$\{(1, 1), (-\alpha, \mathfrak{i}^{-1}), (-\beta, \sqrt{-\beta}^{-1}), (\alpha\beta, \mathfrak{i}^{-1} \sqrt{-\beta}^{-1})\}.$$

According to the proof of Case III in Section 14.2, the images of  $(-\alpha, \mathfrak{i}^{-1})$ ,  $(-\beta, \sqrt{-\beta}^{-1})$  in  $\text{Sp}(W_1, \langle, \rangle_{1,F})$  belong to the centers of  $\text{Sp}(\mathbb{H}(\mathfrak{i}), \langle, \rangle_{1,F}) \times \text{Sp}(\mathbb{H}(\mathfrak{j}), \langle, \rangle_{1,F}) \times \text{Sp}(\mathbb{H}(\mathfrak{l}), \langle, \rangle_{1,F})$ . On the other hand, by observation,  $\mathbb{SL}_1(\mathbb{D}_4)/[\mathbb{SL}_1(\mathbb{D}_4), \mathbb{SL}_1(\mathbb{D}_4)]$  is isomorphic with  $\mathbb{SL}_1(\mathbb{D}_{F(\mathfrak{i})})/[\mathbb{SL}_1(\mathbb{D}_{F(\mathfrak{i})}), \mathbb{SL}_1(\mathbb{D}_{F(\mathfrak{i})})]$ ; taking the set  $\Xi_1$  as in Proposition 14.7, we then have

**Lemma 15.4.** *Let  $\Omega_1$  be the image of the set  $T$  (Proposition 8.10) in  $U(W_1, \langle, \rangle_{1,F})$ . Then*

(1) *The composite map*

$$\Omega_1 \hookrightarrow U(W_1, \langle, \rangle_1) \twoheadrightarrow U(W_1, \langle, \rangle_{1,F})/[U(W_1, \langle, \rangle_{1,F}), U(W_1, \langle, \rangle_{1,F})]$$

*is onto.*

(2) *The image of  $\Omega_1$  in  $\text{Sp}(W_1, \langle, \rangle_{1,F})$  belongs to above  $P(Y)$ .*

*Proof.* We only sketch the proof of the second statement. By the arguments of Section 14.2, an element  $\alpha_1 + A_1\alpha_2 \in \mathbb{SL}_1(F(\mathfrak{i})(A_1))$  should act on  $W_1$  in terms of the matrix with the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & X_{22} & X_{23} \\ 0 & X_{32} & X_{33} \end{pmatrix} \in \text{GL}_3(M_2(F)), \text{ where } \begin{pmatrix} X_{22} & X_{23} \\ X_{32} & X_{33} \end{pmatrix} \text{ is given in Lemma 14.7; the result then follows. } \quad \square$$

Similarly as before, we assume  $\mathfrak{e}_{-1}^{-1} = d_0 - \frac{c_0}{\alpha}\mathfrak{i}$ , and  $\mathfrak{l} = \mathfrak{e}_{-1}^{-1}\mathfrak{k}$ . Let  $\Gamma_1$  be a subgroup of  $\Gamma$  generated by the following elements:

$$(I) \begin{cases} (i) & (a, a, a; a^{-1}) & \text{for all } a \in F^\times, \\ (ii) & (\mathfrak{e}_{-1}\mathfrak{i}, \mathfrak{i}, \mathfrak{i}; \mathfrak{i}^{-1}\mathfrak{e}_{-1}), \\ (iii) & (\mathfrak{e}_{-1}\mathfrak{l}, \mathfrak{e}_{-1}\mathfrak{l}, \mathfrak{l}; \mathfrak{l}^{-1}), \end{cases}$$

if  $-1 \in (F^\times)^2$ ,

$$(II) \begin{cases} (i) & (a, a, a; a^{-1}) & \text{for all } a \in F^\times, \\ (ii) & (\mathfrak{l}, \mathfrak{l}, \mathfrak{e}_{-1}\mathfrak{l}; \mathfrak{l}^{-1}\mathfrak{e}_{-1}^{-1}), \\ (iii) & (\mathfrak{e}_{-1}\mathfrak{l}, \mathfrak{e}_{-1}\mathfrak{l}, \mathfrak{l}; \mathfrak{l}^{-1}), \end{cases}$$

if  $-1 \in N_{F(\mathfrak{i})/F}(F(\mathfrak{i})^\times) \setminus (F^\times)^2$ . Immediately we have

**Lemma 15.5.** (1)  $\Lambda_{\Gamma_1} = F^\times = \Lambda_\Gamma$ .

(2)  $\iota(\Gamma_1) \cap \iota(U(W_1, \langle, \rangle_1) \times U(W_2, \langle, \rangle_2)) = 1$ .

**Lemma 15.6.** *The condition (C4) of Section 10 also holds in this case.*

*Proof.* Recall that  $\Omega_2 = \mathbb{SL}_1(F(\mathfrak{i}))$ , and  $\mathfrak{e}_{-1} \in F(\mathfrak{i})$ , so  $\Omega_2^{\Gamma_1} = \Omega_2$ . For  $\Omega_1$ , the reason is essentially the same.  $\square$

**Proposition 15.7.** *Notations being as in Section 10, under the restriction  $H^2(\iota(\Gamma), \mu_8) \longrightarrow H^2(\iota(\Gamma_1), \mu_8)$ , the image of  $[c]$  is trivial.*

*Proof.* By definition,  $\iota(\Gamma_1)$  belongs to

$$\mathrm{Sp}(\mathbb{H}(\mathfrak{i}), \langle, \rangle_{1,F}) \times \mathrm{Sp}(\mathbb{H}(\mathfrak{j}), \langle, \rangle_{1,F}) \times \mathrm{Sp}(\mathbb{H}(\mathbb{I}), \langle, \rangle_{1,F});$$

we denote its image in the first group by  $\iota(\Gamma_1^{(1)})$ , the second one by  $\iota(\Gamma_1^{(2)})$ , and the third one by  $\iota(\Gamma_1^{(3)})$ . By what we have proved in Lemma 14.3,  $\iota(\Gamma_1^{(1)})$  is spitting. On the other hand  $\iota(\Gamma_1^{(2)})$ ,  $\iota(\Gamma_1^{(3)})$  both belong to  $P(Y_*)$  so the proposition is proved.  $\square$

By Lemma 10.1, as a consequence we obtain

**Proposition 15.8.** *Under the conditions of the beginning, Theorem A holds.*

## 16. THE PROOF OF THE MAIN THEOREM V.

Let  $(H, \langle, \rangle)$  be a right skew hermitian hyperbolic plane over  $\mathbb{H}$  defined as in Section 2. Let  $(W_1 = \mathbb{H}(\mathfrak{i}) \oplus H, \langle, \rangle_1)$  be a right skew hermitian hyperbolic space over  $\mathbb{H}$  of 3 dimension. Let  $(W_2 = \mathbb{H}, \langle, \rangle_2)$  be a left hermitian space over  $\mathbb{H}$  of 1 dimension. Let  $\{1, \mathfrak{i}, \mathfrak{j}, \mathfrak{i}\mathfrak{j} = \mathbb{K} = -\mathfrak{j}\mathfrak{i}\}$  be a standard basis of  $\mathbb{H}$ .

We endow  $W_1$  with the  $F$ -symplectic form  $\langle, \rangle_{1,F} = \mathrm{Trd}(\langle, \rangle_1)$ . Then it can be checked that the canonical mapping

$$\theta : W = W_1 \otimes_{\mathbb{H}} W_2 \longrightarrow W_1 = \mathbb{H}(\mathfrak{i}) \oplus H;$$

$$w_1 \otimes \mathfrak{d} \longmapsto w_1 \mathfrak{d}$$

defines an isometry between  $(W, \langle, \rangle)$  and  $(\mathbb{H}(\mathfrak{i}) \oplus H, \langle, \rangle_{1,F})$ . Let us fix a symplectic basis  $\mathcal{A}_1 = \{-\frac{1}{2\alpha}, -\frac{\mathfrak{j}}{2\alpha\beta}; \mathfrak{i}, \mathbb{K}\}$  of  $\mathbb{H}(\mathfrak{i})$  and a complete polarisation  $H = X \oplus X^*$  of  $H$ . By Propositions 2.12, 3.13, there is a surjective composite map

$$\mathfrak{h} : \frac{\mathbb{H}^\times}{[\mathbb{H}^\times, \mathbb{H}^\times]} \longrightarrow \frac{\mathrm{U}(H, \langle, \rangle_1)}{[\mathrm{U}(H, \langle, \rangle_1), \mathrm{U}(H, \langle, \rangle_1)]} \longrightarrow \frac{\mathrm{U}(W_1, \langle, \rangle_1)}{[\mathrm{U}(W_1, \langle, \rangle_1), \mathrm{U}(W_1, \langle, \rangle_1)]}.$$

**Corollary 16.1.** *Let  $\Omega_1 = \mathbb{H}^\times$ , and  $Y = Y_1 \oplus X^*$ , for an arbitrary Lagrangian subspace  $Y_1$  of  $(\mathbb{H}(\mathfrak{i}), \langle, \rangle_1)$ . Then the image of  $\Omega_1$  in  $\frac{\mathrm{U}(W_1, \langle, \rangle_1)}{[\mathrm{U}(W_1, \langle, \rangle_1), \mathrm{U}(W_1, \langle, \rangle_1)]}$  is full and  $\iota(\Omega_1) \subseteq P(Y)$ .*

**16.1. Case 1:  $\mathfrak{i} = \xi$  and  $F_1 = F(\xi)$ .** We let  $\Omega_2 = \mathbb{SL}_1(F_1)$  with full image in

$$\mathrm{U}(W_2, \langle, \rangle_2) / [\mathrm{U}(W_2, \langle, \rangle_2), \mathrm{U}(W_2, \langle, \rangle_2)].$$

An element  $\mu = a_0 + b_0\mathfrak{i} \in \mathbb{SL}_1(F_1)$ , acts on  $(\mathbb{H}(\mathfrak{i}), \langle, \rangle_{1,F})$  via the matrix  $G_\nu$  defined in Section 11.3. Invoking the results of Proposition 11.10, the image  $\iota(\mathbb{SL}_1(F_1))$  in  $\mathrm{Sp}(\mathbb{H}(\mathfrak{i}), \langle, \rangle_1)$  lies in  $P(Y_2^{(1)})$  for  $Y_2^{(1)} = \mathrm{Span}\{\mathfrak{e}_1 = 1 - \frac{1}{2\alpha}\mathbb{K}, \mathfrak{e}_2 = -2\mathfrak{i} + \mathfrak{j}\}$ . On the other hand, by Proposition 2.12, 2.15,  $P(X^*)$  contains the projection of  $\iota(\Omega_2)$  in  $\mathrm{Sp}(H, \langle, \rangle_{1,F})$ . Obviously one has

**Proposition 16.2.** *Let  $Y = Y_2^{(1)} \oplus X^*$  be a Lagrangian subspace of  $W_1 = \mathbb{H}(\mathfrak{i}) \oplus H$ . Then  $\iota(\Omega_2) \subseteq P(Y)$ .*

**16.2. Case 2:  $\mathfrak{i} = \varpi$  &  $\xi\varpi$ .** Suppose  $\mathfrak{j} = \xi$ , and  $F_2 = F(\mathfrak{j})$ . By Proposition 11.11, we let  $Y_2^{(2)} = \mathrm{Span}\{1, \mathfrak{j}\}$ . Then  $Y_2^{(2)}$  is a Lagrangian subspace of  $(\mathbb{H}(\mathfrak{i}), \langle, \rangle_{1,F})$ , and the image of  $\iota(\mathbb{SL}_1(F_2))$  in  $\mathrm{Sp}(\mathbb{H}(\mathfrak{i}), \langle, \rangle_{1,F})$  belongs to  $P(Y_2^{(2)})$ . Similarly  $P(X^*)$  contains the projection of  $\iota(\Omega_2)$  in  $\mathrm{Sp}(H, \langle, \rangle_{1,F})$ . One also has

**Proposition 16.3.** *Let  $Y = Y_2^{(2)} \oplus X^*$  be a Lagrangian subspace of  $W_1 = \mathbb{H}(\mathfrak{i}) \oplus H$ . Then  $\iota(\Omega_2) \subseteq P(Y)$ .*



16.3. Following the comprehensive discussion in Section 11.5, we define the subgroup  $\Gamma_1$  of  $\Gamma$  generated

$$\begin{aligned}
 & \text{by (I) } \begin{cases} (i) & (a, \text{diag}(a, a); a^{-1}) & \text{for all } a \in F^\times, \\ (ii) & (e_{-1}j, \text{diag}(j, j); j^{-1}) & \text{for some } e_{-1} \in F_1^\times, \text{ with } \text{Nrd}(e_{-1}) = -1, \\ (iii) & (j, \text{diag}(e_{-1}j, e_{-1}j); j^{-1}e_{-1}) & \text{above } e_{-1}, \end{cases} \\
 & \text{if } i = \xi, -1 \in N_{F_1/F}(F_1^\times) \setminus (F^\times)^2, \\
 & \text{by (II) } \begin{cases} (i) & (a, \text{diag}(a, a); a^{-1}) & \text{for all } a \in F^\times, \\ (ii) & (e_{-1}i, \text{diag}(e_{-1}i, e_{-1}i); i^{-1}e_{-1}) & \text{for some } e_{-1} \in F_1^\times, \text{ with } \text{Nrd}(e_{-1}) = -1, \\ (iii) & (j, \text{diag}(e_{-1}j, e_{-1}j); j^{-1}e_{-1}) & \text{above } e_{-1}, \end{cases} \\
 & \text{if } i = \xi, -1 \in (F^\times)^2, \text{ and} \\
 & \text{by (III) } \begin{cases} (i) & (a, \text{diag}(a, a); a^{-1}) & \text{for all } a \in F^\times, \\ (ii) & (i, \text{diag}(i, i); i^{-1}), \\ (iii) & (j, \text{diag}(e_{-1}j, e_{-1}j); j^{-1}e_{-1}) & \text{for some } e_{-1} \in F_2^\times, \text{ satisfying } \text{Nrd}(e_{-1}) = -1, \end{cases} \\
 & \text{if } (i, j) = (\varpi, \xi) \text{ or } (\xi\varpi, \xi). \text{ Immediately, we have}
 \end{aligned}$$

**Lemma 16.4.** (1)  $\Lambda_{\Gamma_1} = F^\times = \Lambda_\Gamma$ .  
 (2)  $\Gamma_1 \cap (\text{U}(W_1, \langle, \rangle_1) \times \text{U}(W_2, \langle, \rangle_2)) = \{(-1, -1), (1, 1)\}$ .

**Lemma 16.5.** The condition (C4) of Section 10 holds.

*Proof.* By Proposition 11.14, the result for  $\Omega_2^{\Gamma_1}$  holds. On the other hand, recall  $\Omega_1 = \mathbb{H}^\times$ , so the action of  $\Omega_1$  on  $\mathbb{H}(i)$  is trivial, and on  $H$  via a diagonal matrix with elements in  $\mathbb{H}^\times$ ; clearly  $\Gamma_1$  stabilizes  $\Omega_1$ .  $\square$

**Proposition 16.6.** Under the restriction  $H^2(\iota(\Gamma), \mu_8) \rightarrow H^2(\iota(\Gamma_1), \mu_8)$ , the image of  $[c]$  is trivial.

*Proof.* First there is a morphism

$$\psi : \text{Sp}(\mathbb{H}(i), \langle, \rangle_{1,F}) \times \text{Sp}(H, \langle, \rangle_{1,F}) \rightarrow \text{Sp}(W_1, \langle, \rangle_{1,F}),$$

which will induce the following morphism on covering groups by [5, Pages 245-246]

$$\bar{\psi} : \overline{\text{Sp}(\mathbb{H}(i), \langle, \rangle_{1,F})} \times \overline{\text{Sp}(H, \langle, \rangle_{1,F})} \rightarrow \overline{\text{Sp}(W_1, \langle, \rangle_{1,F})}.$$

Indeed  $\bar{\psi}$  sends a product of splitting groups to other splitting group. By definition, the map  $\iota$  from  $\Gamma_1$  to  $\text{Sp}(W_1, \langle, \rangle_{1,F})$  should factor through above  $\bar{\psi}$ ; we denote its image in  $\text{Sp}(\mathbb{H}(i), \langle, \rangle_{1,F})$  by  $\Gamma_1^{(1)}$ , and in  $\text{Sp}(H, \langle, \rangle_{1,F})$  by  $\Gamma_1^{(2)}$ . By Proposition 11.15,  $\Gamma_1^{(1)} (\subseteq \overline{\text{Sp}(\mathbb{H}(i), \langle, \rangle_{1,F})})$  is splitting, and by Proposition 12.1,  $\Gamma_1^{(2)} (\subseteq \overline{\text{Sp}(H, \langle, \rangle_{1,F})})$  is also splitting; this completes the proof.  $\square$

Finally we obtain

**Proposition 16.7.** Under conditions of the beginning of this section, Theorem A holds.

## 17. THE PROOF OF THE MAIN THEOREM VI.

In this section we finish proving Theorem A in the general case. The whole process has already done in Section 16. To avoid duplicating work, we only give a sketch of the necessary steps. We will use the notations introduced in Section 1. Recall the main result:

**Theorem 17.1.** The exact sequence

$$1 \rightarrow \mu_8 \rightarrow \bar{\Gamma} \rightarrow \iota(\Gamma) \rightarrow 0$$

is splitting, except when the irreducible dual reductive pair of type I is a symplectic-orthogonal type, and the orthogonal vector space over  $F$  is of odd dimension.

Now let  $W_\nu = W_\nu^0 \oplus H_\nu$  be a Witt's decomposition with  $W_\nu^0$  being its anisotropic subspace, and  $H_\nu \simeq m_\nu H$  being its hyperbolic subspace, as  $\nu$  runs through 1, 2.

**Remark 17.2.** (1) If  $W_1^0 = 0$ , or  $W_2^0 = 0$ , we deduce the result from Section 12.

(2) If  $m_1 = 0 = m_2$ , the result has been verified in Sections 13, 14, 15.

(3) The case  $D = F$ ,  $\{\epsilon_1, \epsilon_2\} = \{\pm 1\}$  has been completely discussed in Corollary 12.2, in what follows we shall exclude that case automatically.

Suppose now  $W_1^0 \neq 0$ ,  $W_2^0 \neq 0$ , and we assume that either  $m_1$  or  $m_2$  is nonzero. In this situation there is a morphism

$$i : \mathrm{Sp}(W_1^0 \otimes_D W_2^0) \times \mathrm{Sp}(W_1^0 \otimes_D H_2) \times \mathrm{Sp}(H_1 \otimes_D W_2^0) \times \mathrm{Sp}(H_1 \otimes_D H_2) \longrightarrow \mathrm{Sp}(W_1 \otimes_D W_2),$$

which induces a morphism on cover groups by [5, Pages 245-246]:

$$\bar{i} : \overline{\mathrm{Sp}(W_1^0 \otimes_D W_2^0)} \times \overline{\mathrm{Sp}(W_1^0 \otimes_D H_2)} \times \overline{\mathrm{Sp}(H_1 \otimes_D W_2^0)} \times \overline{\mathrm{Sp}(H_1 \otimes_D H_2)} \longrightarrow \overline{\mathrm{Sp}(W_1 \otimes_D W_2)}.$$

For these subspaces

$$W_1^0 \otimes_D W_2^0, W_1^0 \otimes_D H_2, H_1 \otimes_D W_2^0, H_1 \otimes_D H_2$$

of  $W_1 \otimes_D W_2$ , we have already defined the suitable pairs

$$(\Gamma^{(0,0)}, \Gamma_1^{(0,0)}), (\Gamma^{(H_1,0)}, \Gamma_1^{(H_1,0)}), (\Gamma^{(0,H_2)}, \Gamma_1^{(0,H_2)}), (\Gamma^{(H_1,H_2)}, \Gamma_1^{(H_1,H_2)})$$

of the subgroups of  $(\mathrm{GU}(W_1^0, \langle, \rangle_1) \times \mathrm{GU}(W_2^0, \langle, \rangle_2)), \dots, (\mathrm{GU}(H_1, \langle, \rangle_1) \times \mathrm{GU}(H_1, \langle, \rangle_2))$  respectively in Sections 12, 13, 14, 15, 16. By Proposition 12.1, we can let  $\Gamma_1$  be a subgroup of  $\Gamma$  consisting of those elements

$$[(g_1^{(0)}, g_1^{(H_1)}), (g_2^{(0)}, g_2^{(H_2)})]$$

such that

- (1)  $(g_1^{(0)}, g_2^{(0)}) \in \Gamma_1^{(0,0)}$ ,
- (2)  $(g_1^{(0)}, g_2^{(H_2)}) \in \Gamma_1^{(0,H_2)}$ ,
- (3)  $(g_1^{(H_1)}, g_2^{(0)}) \in \Gamma_1^{(H_1,0)}$ , and consequently
- (4)  $(g_1^{(H_1)}, g_2^{(H_2)}) \in \Gamma_1^{(H_1,H_2)}$ .

As a consequence we obtain

**Lemma 17.3.** (1)  $\Lambda_\Gamma = \Lambda_{\Gamma_1}$ .

(2)  $\iota(\Gamma_1) \cap \iota(\mathrm{U}(W_1, \langle, \rangle_1) \times \mathrm{U}(W_2, \langle, \rangle_2)) = 1$ .

and

**Proposition 17.4.** Under the restriction  $H^2(\iota(\Gamma), \mu_8) \longrightarrow H^2(\iota(\Gamma_1), \mu_8)$ , the image of  $[c]$  is trivial.

17.1. Suppose now  $m_1 m_2 \neq 0$ . By Propositions 2.12, 3.13, for each  $\nu = 1, 2$ , there exists a surjective composite map

$$\frac{D^\times}{[D^\times, D^\times]} \twoheadrightarrow \frac{\mathrm{U}(H, \langle, \rangle_\nu)}{[\mathrm{U}(H, \langle, \rangle_\nu), \mathrm{U}(H, \langle, \rangle_\nu)]} \twoheadrightarrow \frac{\mathrm{U}(W_\nu, \langle, \rangle_\nu)}{[\mathrm{U}(W_\nu, \langle, \rangle_\nu), \mathrm{U}(W_\nu, \langle, \rangle_\nu)]}.$$

Let  $\Omega_\nu = D^\times$ , viewed as a subgroup of  $\mathrm{U}(W_\nu, \langle, \rangle_\nu)$  in a right way, and let  $Y_\nu = Y_\nu^{(0)} \oplus Y_\nu^{(H_\nu)}$  be a Lagrangian subspace of  $(W_1 \otimes W_2, \langle, \rangle_1 \otimes \tau(\langle, \rangle_2))$  consisting of an arbitrary Lagrangian subspace  $Y_\nu^{(0)}$  of  $W_1^0 \otimes_D W_2^0$ , and  $Y_\nu^{(H_\nu)}$  defined in Proposition 12.1 in each case. By Proposition 12.1 we obtain

**Proposition 17.5.** (1)  $\iota(\Omega_\nu) \subseteq P(Y_\nu)$ .

(2)  $\iota(\Omega_\nu)^{\iota(\Gamma_1)} \subseteq P(Y_\nu)$ .

By Lemma 10.1, Theorem A holds in this special case.

17.2. Without loss of generality, suppose now  $m_1 \neq 0$  and  $m_2 = 0$ . In this case, we let  $\Omega_1 = D^\times$ , and  $Y_1$  be the Lagrangian subspace of  $W_1 \otimes W_2$  defined as before. Nevertheless,  $W_2 = W_2^0$  is an anisotropic  $\epsilon$ -hermitian space over  $D$ . We follow the discussion in Sections 12, 13, 14, 15, 16, and keep defining the distinct set  $\Omega_2$  along that road. With the benefit, we obtain the same result as Proposition 17.5. Without a doubt, Theorem A holds in this case.

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